Carleman estimates for the Zaremba Boundary Condition and Stabilization of Waves.

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Abstract

In this paper, we shall prove a Carleman estimate for the so-called Zaremba problem. Using some techniques of interpolation and spectral estimates, we deduce a result of stabilization for the wave equation by means of a linear Neumann feedback on the boundary. This extends previous results from the literature: indeed, our logarithmic decay result is obtained while the part where the feedback is applied contacts the boundary zone driven by an homogeneous Dirichlet condition. We also derive a controllability result for the heat equation with the Zaremba boundary condition.

Keywords

Carleman estimates, Stabilization of Waves, Zaremba problem, pseudo–differential calculus, controllability.

Contents

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1 Introduction

1.1 General background

We are interested here in the stabilization of the wave equation on a bounded connected regular open set of $\mathbb{R}^d$. Our stabilization will be obtained by means of a feedback on a part of the boundary while the other part of the boundary is submitted to an homogeneous Dirichlet condition.
Since the works of Bardos, Lebeau, Rauch (see [2]), the case of stabilization for the wave equation is well understood (by the so called *Geometric Control Condition*) for Dirichlet or Neumann boundary condition. Indeed, if the part of the boundary driven by the homogeneous Dirichlet condition does not contact the region where the feedback is applied, Lebeau has given a sharp sufficient condition for exponential stabilization of the wave equation (see [20, Théorème 3] and [21]). Moreover, Lebeau and Robbiano (see [23]) have shown that, in the case where the Neumann boundary condition is applied on the entire boundary, a weak condition on the feedback (which does not satisfy *Geometric Control Condition*) provides logarithmic decay of regular solutions.
On the other hand, multiplier techniques (see [15, 8]) give some results of exponential stabilization (even if the part of the boundary driven by the homogeneous Dirichlet condition touches the region where the feedback is applied) but under very strong assumptions on the form of the boundary conditions.
Our goal here is to obtain some stabilization of logarithmic type under weak assumptions for the boundary conditions. More precisely, we will see that, for solutions driven by an homogeneous Dirichlet boundary condition on a part of the boundary and submitted to a feeback of the form

$$\partial_\nu u = -a(x)\partial_t u$$

on the other part of the boundary, where a is some non-trivial non-negative function, their energy with initial data in the domain of $\mathcal{A}^k$ (denoting $\mathcal{A}$ the infinitesimal generator of our evolution equation) decays like $\ln(t)^{-k}$ when t goes to infinity.

To this end, we will need some Carleman estimates for the so-called Zaremba Boundary Problem

$$\begin{cases} \Delta_X u = f & \text{in } X, \\ u = f_0 & \text{on } \partial X_D, \\ \partial_\nu u = f_1 & \text{on } \partial X_N, \end{cases}$$

where X is some regular manifold with boundary ∂X splitted into ∂X_D and ∂X_N and normal vectorfield ν. However, we will mainly tackle some local problem and the following model case (in $\mathbb{R}^n_+$ with the flat metric)

$$\begin{cases} \Delta u = f & \text{in } \{x_n > 0\}, \\ u = f_0 & \text{on } \{x_n = 0, x_1 > 0\}, \\ \partial_{x_n} u = f_1 & \text{on } \{x_n > 0, x_1 > 0\}, \end{cases}$$

should help the reader to understand the main difficulties of this problem.
The Zaremba problem lies in the large class of boundary pseudodifferential operators, studied by many authors. The first one was probably Eskin (see the monograph [9] where pseudodifferential elliptic boundary problems are studied) but then Boutet de Monvel - in [5] - raised the fundamental *transmission condition*. It was shown to play a key role in the resolution of such problems (see the books of Grubb [12] and [13, Chapter 10] where the algebra of pseudodifferential problems is studied in details).

Unfortunately, the Zaremba problem can not be solved by this pseudodifferential calculus. Indeed, its resolution involves a pseudodifferential operator on the boundary that does not satisfy the transmission condition (see [16]). It lies in the general class of operators introduced by Rempel and Schulze in [25] which allow to construct a parametrix for mixed elliptic problems - including the Zaremba problem (see [16] and, more specifically, Section 4.1). However, up our knowledge, a Carleman estimate for the Zaremba problem could not be obtained so far.
Carleman estimates have many applications ranging from the quantification of unique continuation problems, inverse problems, to stabilization issues and control theory (see the survey paper [17] for a general presentation of these topics). This last application was the motivation for the proof of a suitable Carleman estimate (in the papers of either Lebeau and Robbiano [22] or Fursikov and Imanuvilov [10]) and is still animating nowadays a large developpement of Carleman estimates (see e.g. [19, 18] were controllability of parabolic systems with non-smooth coefficients is studied). Finally, we use the approach developped in [21, 23, 7] (also used by other authors - see, e.g., [3]) to deduce our stabilization result. We shall also address a controllability result for the heat equation with the Zaremba boundary condition (based on the approach developped in [22]).

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1.2 Stabilization of Waves

Let Ω be a bounded connected open set of $\mathbb{R}^d$ with $\mathscr{C}^\infty$ boundary $\partial\Omega$. Let also Γ a smooth hypersurface of $\partial\Omega$ which splits the boundary into the two non-empty open sets $\partial\Omega_D$, $\partial\Omega_N$ so that $\partial\Omega = \partial\Omega_D \sqcup \partial\Omega_N \sqcup \Gamma$ (see Figure 1).
We study the decay of the solution of the following problem

$$\begin{cases} (\partial_t^2 - \Delta)u = 0 & \text{in } \Omega \times \mathbb{R}^+, \\ u = 0 & \text{on } \partial\Omega_D \times \mathbb{R}^+, \\ \partial_\nu u + a(x)\partial_t u = 0 & \text{on } \partial\Omega_N \times \mathbb{R}^+, \\ (u, \partial_t u)_{|t=0} = (u_0, u_1) & \text{in } \Omega, \end{cases} \tag{1}$$

where $(u_0, u_1) \in H^1(\Omega) \times L^2(\Omega)$ is such that $u_0 = 0$ in $\partial\Omega_D$ and a is, for some $\rho \in (0,1)$, a non-negative function of $\mathscr{C}^\rho(\partial\Omega_N)$, the space of Hölder continuous functions on $\partial\Omega_N$.

For the sake of simplicity, we here focus on the classical Laplacian Δ but all the results described below remain true with the Laplacian associated to a smooth metric (see Section 4).

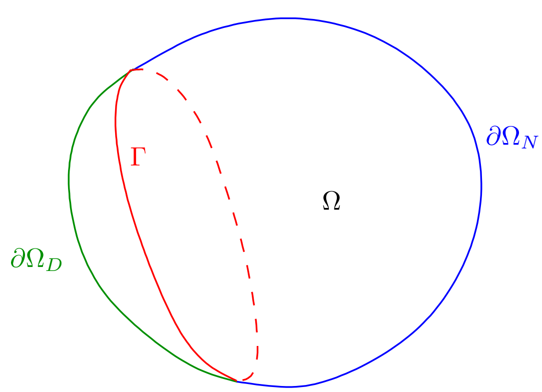

Figure 1: A configuration example.

We denote $H = \{u_0 \in H^1(\Omega); u_0 = 0 \text{ in } \partial\Omega_D\} \times L^2(\Omega)$ and define

$$\mathcal{A} = \begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix}$$

with domain

$$\mathcal{D}(\mathcal{A}) = \{(u_0, u_1) \in H; \Delta u_0 \in L^2(\Omega), u_1 \in H^1(\Omega), u_0 = 0 \text{ on } \partial\Omega_D \text{ and } \partial_\nu u_0 + a(x)u_1 = 0 \text{ on } \partial\Omega_N\}.$$

For any solution u of (1), we define its energy by

$$E(u,t) = \frac{1}{2}\int_\Omega |\partial_t u(x,t)|^2 + |\partial_x u(x,t)|^2 dx$$

where $\partial_x = (\partial_{x_1}, ..., \partial_{x_d})$.
Denoting the resolvent set of $\mathcal{A}$ by

$$\rho(\mathcal{A}) = \{\mu \in \mathbb{C}; \mathcal{A} - \mu I : \mathcal{D}(\mathcal{A}) \to H \text{ is an isomorphism}\},$$

we will establish the following spectral estimate:

Proposition 1.1. *Let $\rho > 1/2$ and $a \in \mathscr{C}^\rho(\partial\Omega_N)$ a non-negative function. If $a \neq 0$ and $a(x) \xrightarrow[x\to\Gamma]{} 0$, then one has $i\mathbb{R} \subset \rho(\mathcal{A})$ and there exists $C > 0$ such that*

$$\forall \lambda \in \mathbb{R},\ \|(\mathcal{A} - i\lambda I)^{-1}\|_{H\to H} \leqslant Ce^{C|\lambda|}.$$

Hence, using an useful result of Burq (see [7, Theorem 3]), we get our logarithmic decay result:

Theorem 1. *Let $\rho > 1/2$ and $a \in \mathscr{C}^\rho(\partial\Omega_N)$ a non-negative function. If $a(x) \xrightarrow[x\to\Gamma]{} 0$ and $a \neq 0$ then, for every $k \geq 1$, there exists $C_k > 0$ such that, for every $(u_0, u_1) \in \mathcal{D}(\mathcal{A}^k)$ the corresponding solution u of (1) satisfies*

$$\forall t \geqslant 0,\ E(u,t)^{1/2} \leqslant \frac{C_k}{\log(2+t)^k}\|(u_0,u_1)\|_{\mathcal{D}(\mathcal{A}^k)}.$$

These results are completely analogous to the ones obtained by Lebeau and Robbiano in [23]. The outline of the proof is also quite similar to the one proposed there except that the situation is a bit different here because of the mixed character of the boundary value problem.
The key point is also to establish some Carleman estimate in a neighborhood of Γ and to obtain some interpolation inequality (see [23, Théorème 3]). This last result concerns an abstract problem derived from the spectral problem.
Defining $X = (-1,1)\times\Omega, \partial X_N = (-1,1)\times\partial\Omega_N, \partial X_D = (-1,1)\times\partial\Omega_D$, we consider the corresponding problem:

$$\begin{cases} \Delta_X v = v_0 & \text{in } X, \\ (\partial_\nu + ia(x)\partial_{x_0})v = v_1 & \text{on } \partial X_N, \\ v = 0 & \text{on } \partial X_D, \end{cases} \tag{2}$$

for some data $v_0 \in L^2(X)$ and $v_1 \in L^2(\partial X_N)$.
If $Y = (-1/2, 1/2) \times \Omega$ and $\partial X_N^\delta = (-1,1) \times \{x \in \partial\Omega_N; a(x) > \delta\}$, we will prove the following interpolation result.

Proposition 1.2. *Let $\rho > 1/2$ and $a \in \mathscr{C}^\rho(\partial\Omega_N)$ a non-negative function. If $a(x) \xrightarrow[x\to\Gamma]{} 0$ and $a \neq 0$, there exists $\delta > 0$, $C > 0$ and $\tau_0 \in (0,1)$ such that for any $\tau \in [0, \tau_0]$ and for any function v solution of (2), the following inequality holds*

$$\|v\|_{H^1(Y)} \leqslant C\left(\|v_0\|_{L^2(X)} + \|v_1\|_{L^2(\partial X_N)} + \|v\|_{L^2(\partial X_N^\delta)} + \|\partial_{x_0}v\|_{L^2(\partial X_N^\delta)}\right)^\tau \|v\|^{1-\tau}_{H^1(X)}.$$

1.3 Carleman estimates for the Zaremba Boundary Condition

We will now present our Carleman estimates and establish first some useful notations. Let $n \geq 2$ be the dimension of the connected manifold X.

1.3.1 Notations

Pseudodifferential operators We use the notation introduced in [22].
First, we shall use in the sequel the notations $\langle\xi\rangle := (1+|\xi|^2)^{\frac{1}{2}}$ and $D_{x_j} = \frac{h}{i}\partial_{x_j}$ for $1 \le j \le n$.
Let us now introduce semi-classical ψDOs. We denote by $S^m(\mathbb{R}^n \times \mathbb{R}^n)$, S^m for short, the space of smooth functions $a(x,\xi,h)$, defined for $h \in (0,h_0]$ for some $h_0 > 0$, that satisfy the following property: for all α, β multi-indices, there exists $C_{\alpha,\beta} \ge 0$, such that

$$\left|\partial_x^\alpha \partial_\xi^\beta a(x,\xi,h)\right| \le C_{\alpha,\beta}\langle\xi\rangle^{m-|\beta|}, \quad x \in \mathbb{R}^n,\ \xi \in \mathbb{R}^n,\ h \in (0,h_0].$$

Then, for all sequences $a_{m-j} \in S^{m-j}$, $j \in \mathbb{N}$, there exists a symbol $a \in S^m$ such that $a \sim \sum_j h^j a_{m-j}$, in the sense that $a - \sum_{j<N} h^j a_{m-j} \in h^N S^{m-N}$ (see for instance [24, Proposition 2.3.2] or [14, Proposition 18.1.3]), with a_m as principal symbol. We define Ψ^m as the space of ψDOs $\mathcal{A} = \mathrm{Op}(a)$, for $a \in S^m$, formally defined by

$$\mathcal{A}\, u(x) = \frac{1}{(2\pi h)^n} \iint e^{i\langle x-t,\xi\rangle/h} a(x,\xi,h)\, u(t)\ dt\ d\xi, \quad u \in \mathscr{S}'(\mathbb{R}^n).$$

We now introduce tangential symbols and associated operators.
We set $x = (x', x_n)$, $x' = (x_1,\dots,x_{n-1})$ and $\xi' = (\xi_1,\dots,\xi_{n-1})$ accordingly. We denote by $S^m_{\mathcal{T}}(\mathbb{R}^n \times \mathbb{R}^{n-1})$, $S^m_{\mathcal{T}}$ for short, the space of smooth functions $b(x,\xi',h)$, defined for $h \in (0,h_0]$ for some $h_0 > 0$, that satisfy the following property: for all α, β multi-indices, there exists $C_{\alpha,\beta} \ge 0$, such that

$$\left|\partial_x^\alpha \partial_{\xi'}^\beta b(x,\xi',h)\right| \le C_{\alpha,\beta}\langle\xi'\rangle^{m-|\beta|}, \quad x \in \mathbb{R}^n,\ \xi' \in \mathbb{R}^{n-1},\ h \in (0,h_0].$$

As above, for all sequences $b_{m-j} \in S^{m-j}_{\mathcal{T}}$, $j \in \mathbb{N}$, there exists a symbol $b \in S^m_{\mathcal{T}}$ such that $b \sim \sum_j h^j b_{m-j}$, in the sense that $b - \sum_{j<N} h^j b_{m-j} \in h^N S^{m-N}_{\mathcal{T}}$, with b_m as principal symbol. We define $\Psi^m_{\mathcal{T}}$ as the space of tangential ψDOs $B = \mathrm{op}(b)$ (observe the notation we adopt is different from above to avoid confusion), for $b \in S^m_{\mathcal{T}}$, formally defined by

$$B\, u(x) = \frac{1}{(2\pi h)^{n-1}} \iint e^{i\langle x'-t',\xi'\rangle/h} b(x,\xi',h)\, u(t',x_n)\ dt'\ d\xi', \quad u \in \mathscr{S}'(\mathbb{R}^n).$$

We shall also denote the principal symbol b_m by $\sigma(B)$.

Different norms We use L^2 and H^s_{sc} semi-classical norms on $\mathbb{R}^n$, on $\{x_n > 0\}$, on $\{x_n = 0\}$ and on $\{x_n = 0,\ \pm x_1 > 0\}$. We recall that, in this paper, we use the usual semi-classical notations, namely $D_{x_j} = \frac{h}{i}\partial_{x_j}$, and the symbols are quantified in semi-classical sense. In particular all the norms depend on h.

To distinguish these different norms, we denote by

$$\|u\|^2 = \int_{\mathbb{R}^n} |u(x)|^2 dx, \quad \|u\|_s = \|\,\mathrm{Op}(\langle\xi\rangle^s)u\|$$

and

$$\|u\|^2_{L^2(x_n>0)} = \int_{\{x\in\mathbb{R}^n,\ x_n>0\}} |u(x)|^2 dx, \quad \|u\|^2_{H^1_{sc}(x_n>0)} = \|u\|^2_{L^2(x_n>0)} + \sum_{j=1}^n \|D_{x_j}u\|^2_{L^2(x_n>0)}.$$

Finally, on $x_n = 0$, we use the norms

$$|v|^2 = \int_{\mathbb{R}^{n-1}} |v(x')|^2 dx', \quad |v|_s = |\,\mathrm{op}(\langle\xi'\rangle^s)v|,$$

and the space $H^s_{sc}(\pm x_1 > 0)$ of the restrictions of $H^s_{sc}(\mathbb{R}^{n-1})$ functions equipped with the norm

$$|v|_{H^s_{sc}(\pm x_1>0)} = \inf_{\substack{w\in H^s_{sc}(\mathbb{R}^{n-1})\\ w_{|\pm x_1>0}=v}} |\,\mathrm{op}(\langle\xi'\rangle^s)w|.$$

In particular for $v \in H^s(\mathbb{R}^{n-1})$, we have

$$|v_{|\pm x_1>0}|_{H^s_{sc}(\pm x_1>0)} \leq |v|_s$$

and we write, when there is no ambiguity, $|v|_{H^s_{sc}(\pm x_1>0)}$ instead of $|v_{|\pm x_1>0}|_{H^s_{sc}(\pm x_1>0)}$.

1.3.2 Carleman estimate

We now detail the local Carleman estimate obtained for the Zaremba boundary problem.

Let $B_\kappa = \{x \in \mathbb{R}^n; |x| \leqslant \kappa\}$ and P a differential operator whose form is

$$P = -\partial_{x_n}^2 + R\left(x, \frac{1}{i}\partial_{x'}\right)$$

where $\partial_{x'} = (\partial_{x_1}, \dots, \partial_{x_{n-1}})$ and the symbol $r(x,\xi')$ of R is real, homogeneous of degree 2 in ξ' and satisfies

$$\begin{cases} \exists c>0; \forall (x,\xi') \in B_\kappa \times \mathbb{R}^{n-1},\ r(x,\xi') \geq c|\xi'|^2, \\ \forall \xi' \in \mathbb{R}^{n-1}, r(0,\xi') = |\xi'|^2. \end{cases}$$

As usual in the context of Carleman estimates, we define the conjugate $P_\varphi = h^2 e^{\varphi/h} \circ P \circ e^{-\varphi/h}$ for φ any real-valued $\mathscr{C}^\infty$ function. Since

$$P_\varphi = h^2\left(\frac{1}{i}\partial_{x_n} + \frac{i}{h}\partial_{x_n}\varphi\right)^2 + h^2 R\left(x, \frac{1}{i}\partial_{x'} + \frac{i}{h}\partial_{x'}\varphi\right),$$

the corresponding semi-classical principal symbol satisfies

$$p_\varphi(x,\xi) = (\xi_n + i\partial_{x_n}\varphi(x))^2 + r(x, \xi' + i\partial_{x'}\varphi(x)).$$

We assume that φ is such that, for some $\kappa_0 > 0$,

$$\forall x \in B_{\kappa_0}, \frac{\partial \varphi}{\partial x_n}(x) \neq 0 \tag{3}$$

and that Hörmander pseudo-convexity hypothesis (see [14, Paragraph 28.2, 28.3]) holds for P on B_{κ_0}

$$\forall (x,\xi) \in B_{\kappa_0} \times \mathbb{R}^n, p_\varphi(x,\xi) = 0 \Rightarrow \{\operatorname{Re} p_\varphi, \operatorname{Im} p_\varphi\}(x,\xi) > 0, \tag{4}$$

where the usual Poisson bracket is defined, for p, q smooth functions, by

$$\{p,q\}(x,\xi) = (\partial_\xi p . \partial_x q - \partial_x p . \partial_\xi q)(x,\xi).$$

Remark 1. For instance in the model case $P = -\Delta$, we can choose $\varphi(x_n) = x_n + \frac{a}{2}x_n^2$. We have indeed $p_\varphi(x,\xi) = (\xi_n + i(1+ax_n))^2 + |\xi'|^2$ thus $\{\operatorname{Re} p_\varphi, \operatorname{Im} p_\varphi\}(x,\xi) = 4a(\xi_n^2 + (1+ax_n)^2) > 0$ if x_n is small enough.

In the general case, changing φ into $e^{\beta\varphi}$ for $\beta > 0$ large enough, hypothesis (4) can be satisfied (see [14, Proposition 28.3.3] or [22, Proof of Lemma 3, page 352]).

Our local Carleman estimate for the Zaremba Boundary Condition can now be stated in the following form.

Theorem 2. *There exists $\varepsilon > 0$ such that if φ satisfies*

$$\left(\frac{\partial \varphi}{\partial x_n} > 0 \text{ on } \{x_n = 0\} \cap B_{\kappa_0}\right) \text{ and } |\partial_{x'}\varphi(0)| \leq \varepsilon \partial_{x_n}\varphi(0)$$

and (3), (4) *hold then, there exists* $\kappa \in (0, \kappa_0]$ *and* $C, h_0 > 0$*, such that, for any* $h \in (0, h_0)$*,* $g_0 \in H^{1/2}(x_1 > 0)$*,* $g_1 \in H^{-1/2}(x_1 < 0)$ *and any* $g \in H^1(\mathbb{R}^n)$ *supported in* B_κ *which satisfies*

$$P(g) \in L^2(\mathbb{R}^n) \text{ and } \begin{cases} g = g_0 & \text{if } x_n = 0 \text{ and } x_1 > 0, \\ \partial_{x_n} g = g_1 & \text{if } x_n = 0 \text{ and } x_1 < 0, \end{cases}$$

the following inequality holds:

$$\|ge^{\varphi/h}\|_{H^1_{sc}(x_n>0)} + |ge^{\varphi/h}|_{1/2} + |h(\partial_{x_n} g)e^{\varphi/h}|_{-1/2}$$
$$\leq C\left(h^{-1/2}\|h^2 P(g)e^{\varphi/h}\|_{L^2(x_n>0)} + |g_0 e^{\varphi/h}|_{H^{1/2}_{sc}(x_1>0)} + |hg_1 e^{\varphi/h}|_{H^{-1/2}_{sc}(x_1<0)}\right).$$

Remark 2. The estimate in the theorem, except for the boundary terms, is the usual Carleman estimate. Let us also remind that all the norms are semi-classical: in particular $\|ge^{\varphi/h}\|_{H^1_{sc}(x_n>0)}$ is equivalent to $h\|e^{\varphi/h}\partial_x g\|_{L^2(x_n>0)} + \|e^{\varphi/h} g\|_{L^2(x_n>0)}$. For the other norms, we refer the reader to the definitions in paragraph 1.3.1.

Remark 3. The norms $|.|_{1/2}$ and $|.|_{-1/2}$ on the boundary $x_n = 0$ of the left hand side of this inequality cannot be replaced by the norms $|.|_1$ and $|.|$ (provided that the data g_0, g_1 are estimated in the spaces $H^1(x_1 > 0)$ and $L^2(x_1 < 0)$).
Indeed, in the special case where $P = -\Delta$, it is well-known that the variational solution of the boundary value problem

$$\begin{cases} -\Delta u = f & \text{in } X, \\ u = 0 & \text{on } \partial X_D, \\ \partial_\nu u = 0 & \text{on } \partial X_N, \end{cases}$$

may be, even for smooth data f, such that $\partial_\nu u \notin L^2(\partial X)$.
We refer the reader to the famous two-dimensionnal conterexample of Shamir (see [26]) where one consider, in polar coordinates, the sets

$$X = \{(r, \theta); r \in (0,1), \theta \in (0, \pi)\}, \quad \partial X_N = \{(r, \pi); r \in (0,1)\}, \quad \partial X_D = \partial X \setminus \overline{\partial X_N}.$$

and the function

$$u(r, \theta) = \phi(r) r^{1/2} \sin\left(\frac{\theta}{2}\right)$$

with $\phi \in \mathscr{C}^\infty([0,1])$ a cut-off function such that $\phi = 1$ in some neighborhood of 0 and $\mathrm{supp}(\phi) \subset [0,1)$.

The paper is structured as follows: our proof of the main Carleman estimate (Theorem 2) is divided into the three subsections of Section 2. This will allow us to deduce the interpolation inequality of Proposition 1.2 and finally Theorem 1 in Section 3. In Section 4, we conclude by some comments on the geometry and sketch a proof of controllability of the heat equation with the Zaremba boundary condition.

2 Proof of Theorem 2

We first recall some well-known facts about pseudodifferential operators. We refer the reader to [24]. For simplicity, we write in all this section $\|\cdot\|_{H^s(x_n>0)}$ instead of $\|\cdot\|_{H^s_{sc}(x_n>0)}$. Note that there will be no confusion as we do not use the classical norm on $H^s(x_n > 0)$.

Composition formula. If $a \in S^m$, $b \in S^{m'}$ then $\mathrm{Op}(a) \circ \mathrm{Op}(b) = \mathrm{Op}(c)$ for $c \in S^{m+m'}$ given by

$$c(x, \xi, h) = \left(\sum_{|\alpha| \leqslant N} \frac{(h/i)^{|\alpha|}}{\alpha!} \partial_\xi^\alpha a \partial_x^\alpha b\right)(x, \xi, h) + h^{N+1} R(x, \xi, h)$$

where

$$R(x,\xi,h)=\frac{N+1}{(2\pi h)^n}\int_0^1(1-t)^N\sum_{|\alpha|=N+1}\frac{1}{i^{|\alpha|}\alpha!}\int_{\mathbb{R}^{2n}}e^{-iz.\zeta/h}\partial_\xi^\alpha a(x,\xi+\zeta,h)\partial_x^\alpha b(x+tz,\xi,h)dzd\zeta dt.$$

We will also use the composition formula for tangential operators, which is completely analogous. In the sequel, we will also need the following straightforward result.

Lemma 2.1. *Let $a\in S_T^m$. Then*

$$[D_{x_n},\mathrm{op}(a)]=\frac{h}{i}\,\mathrm{op}(\partial_{x_n}a).$$

Next, we use the same notations as in [23] and put

$$p_\varphi(x,\xi)=\xi_n^2+2i(\partial_{x_n}\varphi)\xi_n+q_2(x,\xi')+2iq_1(x,\xi') \tag{5}$$

$$\text{where } q_2(x,\xi')=-(\partial_{x_n}\varphi)^2+r(x,\xi')-r(x,\partial_{x'}\varphi(x)) \text{ and } q_1(x,\xi')=\tilde{r}(x,\partial_{x'}\varphi(x),\xi').$$

Here we denote by $\tilde{r}(x,.,.)$ the bilinear form associated to $r(x,.)$ (i.e. such that $r(x,\xi')=\tilde{r}(x,\xi',\xi')$ for all $\xi'\in\mathbb{R}^{n-1}$).
We also define

$$\mu(x,\xi'):=q_2(x,\xi')+\frac{q_1(x,\xi')^2}{(\partial_{x_n}\varphi(x))^2}.$$

The sign of μ is of great importance to localize the roots of p_φ in ξ_n. We may explain this from the model case presented in the introduction. In this framework, one has $P=-\Delta$ and we may choose $\varphi=\varphi(x_n)$ (more precisely of the form $\varphi(x_n)=x_n+ax_n^2/2$ for some $a>0$) so that

$$p_\varphi(x,\xi')=(\xi_n+i\partial_{x_n}\varphi(x))^2+|\xi'|^2,$$

$$q_2(x,\xi')=-(\partial_{x_n}\varphi(x))^2+|\xi'|^2,\quad q_1(x,\xi')=0\;\text{ and }\;\mu(x,\xi')=|\xi'|^2-(\partial_{x_n}\varphi(x))^2.$$

Moreover, the roots of p_φ in ξ_n are given by

$$\rho_1(x,\xi')=-i(\partial_{x_n}\varphi(x)-|\xi'|),\quad \rho_2(x,\xi')=-i(\partial_{x_n}\varphi(x)+|\xi'|)$$

and satisfy

$$\mu(x,\xi')>0\Rightarrow \mathrm{Im}(\rho_1(x,\xi'))>0>\mathrm{Im}(\rho_2(x,\xi'))$$

whereas

$$\mu(x,\xi')<0\Rightarrow \mathrm{Im}(\rho_{1,2}(x,\xi'))<0.$$

In the microlocal zone $\mu<0$, the operator p_φ is elliptic and since its roots in ξ_n have negative imaginary part, one will be able to estimate directly the traces of g in terms of the interior data $P(g)$. On the contrary, in the microlocal zone $\mu>0$, even is p_φ is elliptic, only one of its root in ξ_n has negative imaginary part and elliptic estimates would only get an equation on the traces of g. In our general framework, we prove several analogous properties presented in Lemma B.1 (which are very close to the ones of [23, Lemme 3]) and the case $\mu>0$ will in fact be treated in section 2.2.

Our proof of Theorem 2 is consequently divided in two main parts. In the first one, we establish a microlocal Carleman inequality concentrated where $\mu<0$ and, in the second one, we focus on the microlocal region $\mu>-(\partial_{x_n}\varphi)^2$. We will finally gather the results of these two parts in a short concluding section.

Notations: In the sequel, we set, for w a function defined on $\mathbb{R}^n$,

$$\underline{w}=\begin{cases} w & \text{if } x_n>0,\\ 0 & \text{if } x_n<0.\end{cases}$$

We also denote, for $z\in\mathbb{C}/\mathbb{R}_-$ and $s\in\mathbb{R}$,

$$z^s=\exp\left(s\log(z)\right)$$

where log is defined as an holomorphic function on $\mathbb{C}\backslash\mathbb{R}_-$. Moreover, we use the notation $\sqrt{z}:=z^{1/2}$.

2.1 Estimates in zone $\mu < 0$

We remind that we have denoted $v = e^{\varphi/h}g$. We also define the set

$$\mathcal{E}_\alpha = \{(x,\xi') \in \mathbb{R}^n \times \mathbb{R}^{n-1},\ \mu(x,\xi') \leq -\alpha(\partial_{x_n}\varphi)^2\}$$

where $\alpha > 0$ is a sufficiently small parameter to be fixed later.

The proof we give essentially follows that of Lemma 4 in [23] and Proposition 2.2 in [19].

Let χ_- supported in $\mathcal{E}_{2\alpha}$ and satisfying $\chi_- = 1$ in a neighborhood of $\mathcal{E}_{3\alpha}$. Obviously $\chi_- \in S^0_\mathcal{T}$ because $\chi_- = 0$ when $|\xi'|$ is large enough. If $u = \mathrm{op}(\chi_-)v$, one has

$$\begin{aligned} &P_\varphi u = \mathrm{op}(\chi_-)P_\varphi v + [P_\varphi, \mathrm{op}(\chi_-)]v = f_1 \text{ where} \\ &\|f_1\|_{L^2(x_n>0)} \leq C\|P_\varphi v\|_{L^2(x_n>0)} + Ch\|v\|_{H^1(x_n>0)}. \end{aligned} \tag{6}$$

Denoting $\delta^{(j)} = (d/dx_n)^j\delta_{|x_n=0}$, straightforward computation show that we have

$$P_\varphi \underline{u} = \underline{f_1} - h^2\gamma_0(u)\delta' - ih(\gamma_1(u) + 2i\partial_{x_n}\varphi(x',0)\gamma_0(u))\delta \tag{7}$$

where $\gamma_0(u) := u_{|x_n=0^+}$ and $\gamma_1(u) := D_{x_n}u_{|x_n=0^+} = -ih\partial_{x_n}u_{|x_n=0^+}$ are the first semi-classical traces. We now construct a local parametrix for P_φ.
Let $\chi(x,\xi) \in S^0$ such that $\chi = 1$ for sufficiently large $|\xi|$ as well as in a neighborhood of $\mathrm{supp}(\chi_-)$ with moreover

$$\mathrm{supp}(\chi) \cap p_\varphi^{-1}(\{0\}) = \emptyset.$$

Note that it is indeed possible because the real null set $p_\varphi^{-1}(\{0\})$ is bounded in ξ and, using Lemma B.1, the roots of p_φ in ξ_n are not real.
We define

$$e_0(x,\xi) = \frac{\chi(x,\xi)}{p_\varphi(x,\xi)} \in S^{-2}.$$

One may find $e_1 \in S^{-3}$ such that $E = \mathrm{Op}(e_0 + he_1)$ satisfies, for some $R_2 \in S^{-2}$,

$$E \circ P_\varphi = \mathrm{Op}(\chi) + h^2R_2.$$

Indeed, by symbolic calculus, one may verify that $e_1 = \chi\frac{\partial_x p_\varphi.\partial_\xi p_\varphi}{p_\varphi^3}$. In the sequel, we shall denote $e := e_0 + he_1$.
We set the new quantities

$$w_1 := \gamma_0(u),\ w_0 := \gamma_1(u) + 2i\partial_{x_n}\varphi(x',0)\gamma_0(u) \tag{8}$$

and we apply our parametrix E to the equation (7) which may be written now in the form

$$P_\varphi\underline{u} = \underline{f_1} + \frac{h}{i}w_0\delta - h^2w_1\delta'.$$

One computes the action of E on w_0 and w_1 and finds

$$E\left(\frac{h}{i}w_0\delta\right)(x',x_n) = \frac{1}{(2\pi h)^{n-1}}\iint e^{i(x'-y').\xi'/h}\hat{t_0}(x_n,x',\xi')w_0(y')dy'd\xi',$$

$$E\left(-h^2w_1\delta'\right)(x',x_n) = \frac{1}{(2\pi h)^{n-1}}\iint e^{i(x'-y').\xi'/h}\hat{t_1}(x_n,x',\xi')w_1(y')dy'd\xi',$$

where

$$\hat{t_0}(x_n,x',\xi') = \frac{1}{2i\pi}\int_\mathbb{R} e^{ix_n\xi_n/h}e(x,\xi)d\xi_n,$$

$$\hat{t_1}(x_n,x',\xi') = \frac{1}{2i\pi}\int_\mathbb{R} e^{ix_n\xi_n/h}\xi_n e(x,\xi)d\xi_n.$$

We note that the integral defining $\hat{t_0}$ is absolutely converging but that the integral defining $\hat{t_1}$ has to be understood is the sense of the oscillatory integrals (see for instance [14, Section 7.8]).
Using the fact that $e(x,\xi',\xi_n)$ is holomorphic for large $|\xi_n|$ and actually a rational function with respect to ξ_n, we can change the contour $\mathbb{R}$ into the contour defined by $\gamma = [-C\langle\xi'\rangle, C\langle\xi'\rangle] \cup \{\xi_n \in \mathbb{C}; |\xi_n| = C\langle\xi'\rangle, \mathrm{Im}(\xi_n) > 0\}$ oriented counterclockwise where $C > 0$ is chosen sufficiently large so that $\chi = 1$ if $|\xi_n| \geq C\langle\xi'\rangle$.

Doing so, we get

$$\underline{u} = E(\underline{f_1}) + T_0 w_0 + T_1 w_1 + r_1$$

where

$$r_1 = (I - \mathrm{Op}(\chi))\underline{u} + h^2 R_2 \underline{u} \tag{9}$$

and, if $j = 0, 1$ and $x_n > 0$, the tangential operators T_j of symbols

$$\hat{t_j}(x,\xi') = \frac{1}{2i\pi}\int_\gamma e^{ix_n(\xi_n/h)} e(x', x_n, \xi', \xi_n)\xi_n^j d\xi_n, \tag{10}$$

The symbols $1-\chi$ and χ_- are not in the same symbol class but it is known (see Lebeau-Robbiano [22] and Le Rousseau-Robbiano [19, Lemma 2.2]) that, since $\mathrm{supp}(1-\chi) \cap \mathrm{supp}\,\chi_- = \emptyset$, we have

$$(I - \mathrm{Op}(\chi))\,\mathrm{op}(\chi_-) \in \bigcap_{N\in\mathbb{N}} h^N \Psi^{-N}.$$

Consequently, recalling (9), one has the estimate

$$\|r_1\|_2 \leqslant Ch\|\underline{v}\| = Ch\|v\|_{L^2(x_n>0)} \leq Ch\|v\|_{H^1(x_n>0)}. \tag{11}$$

We now choose $\chi_1(x,\xi') \in S^0_{\mathcal{T}}$ so that $\mathrm{supp}(\chi_-) \subset \{\chi_1 = 1\}$, χ_1 is supported in $\mathcal{E}_\alpha$ and $\chi = 1$ in a neighborhood of $\mathrm{supp}(\chi_1)$.
We set $t_j = \hat{t_j}\chi_1$ for $j = 0, 1$ which allows us to get

$$\underline{u} = E(\underline{f_1}) + \mathrm{op}(t_0)w_0 + \mathrm{op}(t_1)w_1 + r_1 + r_2, \tag{12}$$

where

$$r_2 = \mathrm{op}((1-\chi_1)\hat{t_0})w_0 + \mathrm{op}((1-\chi_1)\hat{t_1})w_1.$$

One now notes that $|p_\varphi(x,\xi)| \geqslant c\langle\xi\rangle^2$ on $\mathrm{supp}(\chi)$. Consequently, one obtains

$$\|E(\underline{f_1})\|_1 \leqslant C\|\underline{f_1}\| = C\|f_1\|_{L^2(x_n>0)}. \tag{13}$$

Moreover, using (10), one may obtain

$$\forall l \in \mathbb{N}, \alpha \in \mathbb{N}^{n-1}, \beta \in \mathbb{N}^{n-1},\ |\partial^l_{x_n}\partial^\alpha_{x'}\partial^\beta_{\xi'}\hat{t_j}| \leq Ch^{-l}\langle\xi'\rangle^{-1+j+l-|\beta|}.$$

Consequently, noting that r_2 does not involve derivations with respect to x_n and that $\mathrm{supp}(1-\chi_1) \cap \mathrm{supp}(\chi_{-|x_n=0}) = \emptyset$, one obtains

$$\|r_2\|_1 \leqslant Ch(\|v\|_{H^1(x_n>0)} + |D_{x_n} v_{|x_n=0}|_{-1/2}) \tag{14}$$

from the composition of tangential operators and using the following trace formula (see [23, page 486])

$$|\psi_{|x_n=0}| \leqslant Ch^{-1/2}\|\psi\|_{H^1(x_n>0)}.$$

Regarding the two last terms, we use that $\mu(x,\xi') < 0$ for $(x,\xi') \in \mathrm{supp}(\chi_1)$. Hence, by Lemma B.1, $p_\varphi(x,\xi',\xi_n)^{-1}$ is an holomorphic function of ξ_n on $\{\mathrm{Im}(\xi_n) \geq 0\}$ for $(x,\xi') \in \mathrm{supp}(\chi_1)$.
Recalling the form of e, one consequently has, for $j = 0, 1$,

$$\begin{aligned}(t_j)(x,\xi') &= \frac{1}{2i\pi}\chi_1(x,\xi')\left(\int_\gamma e^{ix_n\xi_n/h}\frac{\xi_n^j}{p_\varphi(x,\xi',\xi_n)}d\xi_n + h\int_\gamma e^{ix_n\xi_n/h}\frac{\xi_n^j(\partial_x p_\varphi.\partial_\xi p_\varphi)(x,\xi',\xi_n)}{p_\varphi^3(x,\xi',\xi_n)}d\xi_n\right)\\ &= 0.\end{aligned}$$

We shall now address the traces terms. We take the first two traces at $x_n = 0^+$ of (12) which consequently gives, for $j = 0, 1$,

$$\gamma_j(u) = \gamma_j(E(\underline{f_1})) + \gamma_j(r_1) + \gamma_j(r_2).$$

Summing up equations (11), (13) and (14), one now deduces by trace formula

$$h^{1/2}|\gamma_0(u)|_{1/2} + h^{1/2}|\gamma_1(u)|_{-1/2} \le C(\|f_1\|_{L^2(x_n>0)} + h\|v\|_{H^1(x_n>0)} + h|D_{x_n}v_{|x_n=0}|_{-1/2}).$$

Using (12) again, one may deduce

$$\|u\|_{H^1(x_n>0)} + h^{1/2}|\gamma_0(u)|_{1/2} + h^{1/2}|\gamma_1(u)|_{-1/2} \le C(\|f_1\|_{L^2(x_n>0)} + h\|v\|_{H^1(x_n>0)} + h|D_{x_n}v_{|x_n=0}|_{-1/2}).$$

We finally come back to the original unknowns. One has

$$\gamma_0(u) = \mathrm{op}(\chi_-)v_{|x_n=0} \quad \text{and} \quad \mathrm{op}(\chi_-)D_{x_n}v_{|x_n=0} = \gamma_1(u) - [D_{x_n}, \mathrm{op}(\chi_-)]v_{|x_n=0}$$

which, using Lemma 2.1 and (6), allows us to get

$$\begin{aligned} \|\mathrm{op}(\chi_-)v\|_{H^1(x_n>0)} + h^{1/2}|\mathrm{op}(\chi_-)v_{|x_n=0}|_{1/2} + h^{1/2}|\mathrm{op}(\chi_-)D_{x_n}v_{|x_n=0}|_{-1/2} \\ \le C(\|P_\varphi v\|_{L^2(x_n>0)} + h\|v\|_{H^1(x_n>0)} + h|D_{x_n}v_{|x_n=0}|_{-1/2}). \end{aligned} \tag{15}$$

2.2 Estimates in zone $\mu > -(\partial_{x_n}\varphi)^2$

We denote by $v = e^{\varphi/h}g$ and

$$\begin{aligned} v_0 &= v_{|x_n=0} - \left(e^{\varphi/h}\right)_{|x_n=0} g_0 \in H^{1/2}(x_n = 0), \\ v_1 &= (D_{x_n}v)_{|x_n=0} + i(\partial_{x_n}\varphi)_{|x_n=0}v_0 + \left(e^{\varphi/h}\right)_{|x_n=0}(ihg_1 + i(\partial_{x_n}\varphi)_{|x_n=0}g_0) \in H^{-1/2}(x_n = 0). \end{aligned} \tag{16}$$

We have $\operatorname{supp} v_0 \subset \{x' \in \mathbb{R}^{n-1},\ x_1 \ge 0\}$ and $\operatorname{supp} v_1 \subset \{x' \in \mathbb{R}^{n-1},\ x_1 \le 0\}$. We consider v_0 and v_1 as unknown in the problem and in the sequel the goal is to obtain an equation on v_0 and v_1.
The boundary conditions take the following form

$$\begin{aligned} v_{|x_n=0} &= v_0 + G_0, \\ (D_{x_n}v)_{|x_n=0} &= v_1 - i(\partial_{x_n}\varphi)_{|x_n=0}v_0 + G_1, \end{aligned} \tag{17}$$

where, following (16), we have

$$\begin{aligned} |G_0|_{1/2} &\le |e^{\varphi/h}g_0|_{1/2}, \\ |G_1|_{-1/2} &\le h|e^{\varphi/h}g_1|_{-1/2} + C|e^{\varphi/h}g_0|_{1/2}. \end{aligned} \tag{18}$$

We remark that if g_0 is fixed on $x_1 < 0$ and g_1 is fixed on $x_1 > 0$ we can extend g_0 and g_1 on $\mathbb{R}^{n-1}$ such that

$$|e^{\varphi/h}g_0|_{H^{1/2}(x_1<0)} \le |e^{\varphi/h}g_0|_{1/2} \le 2|e^{\varphi/h}g_0|_{H^{1/2}(x_1<0)}$$

and

$$|e^{\varphi/h}g_1|_{H^{-1/2}(x_1>0)} \le |e^{\varphi/h}g_1|_{-1/2} \le 2|e^{\varphi/h}g_1|_{H^{-1/2}(x_1>0)}.$$

These extensions depend on h.

Let

$$\mathcal{F}_\alpha = \{((x,\xi') \in \mathbb{R}^n \times \mathbb{R}^{n-1},\ \mu(x,\xi') \ge -(1-\alpha)(\partial_{x_n}\varphi)^2\}$$

where α is small enough.
Let $\chi_+(x,\xi')$ supported in $\mathcal{F}_{2\alpha}$ and satisfying $\chi_+ = 1$ in a neighborhood of $\mathcal{F}_{3\alpha}$. Obviously $\chi_+ \in S^0_{\mathcal{T}}$ because $\chi_+ = 1$ when $|\xi'|$ is large enough.
Let $u = \mathrm{op}(\chi_+)v$. We have

$$\begin{aligned} &P_\varphi u = \mathrm{op}(\chi_+)P_\varphi v + [P_\varphi, \mathrm{op}(\chi_+)]v = f_1 \text{ where} \\ &\|f_1\|_{L^2(x_n>0)} \le C\|P_\varphi v\|_{L^2(x_n>0)} + Ch\|v\|_{H^1(x_n>0)}. \end{aligned} \tag{19}$$

Let χ_1 supported in $\mathcal{F}_\alpha$ such that $\chi_1 = 1$ on a neighborhood of $\mathcal{F}_{2\alpha}$, in particular on $\operatorname{supp}\chi_+$. When the roots ρ_j are well defined (see the Lemma B.1), we have by (5) $\rho_1 + \rho_2 = -2i(\partial_{x_n}\varphi)$ and $\rho_1\rho_2 = q_2 + 2iq_1$. Hence, we obtain

$$\begin{aligned}(D_{x_n} - \operatorname{op}(\rho_2\chi_1))(D_{x_n} - \operatorname{op}(\rho_1\chi_1))u &= D_{x_n}^2 u - D_{x_n}\operatorname{op}(\rho_1\chi_1)u - \operatorname{op}(\rho_2\chi_1)D_{x_n}u \\ &\quad + \operatorname{op}(\rho_2\chi_1)\operatorname{op}(\rho_1\chi_1)u \\ &= D_{x_n}^2 u + \operatorname{op}(2i(\partial_{x_n}\varphi)\chi_1)D_{x_n}u + \operatorname{op}((q_2+2iq_1)\chi_1^2)u \\ &\quad - [D_{x_n}, \operatorname{op}(\rho_1\chi_1)]u + \operatorname{op}(R_1)u \end{aligned} \tag{20}$$

where $R_1 \in hS^1_{\mathcal{T}}$ is given by symbolic calculus.
By Lemma 2.1, the symbol of $[D_{x_n}, \operatorname{op}(\rho_1\chi_1)]$ belongs to $hS^1_{\mathcal{T}}$. Thus, we have

$$(D_{x_n} - \operatorname{op}(\rho_2\chi_1))(D_{x_n} - \operatorname{op}(\rho_1\chi_1))u = f_2 \tag{21}$$

where

$$\begin{aligned} f_2 =& P_\varphi u - \operatorname{op}(2i(\partial_{x_n}\varphi)(1-\chi_1))D_{x_n}\operatorname{op}(\chi_+)v - \operatorname{op}((q_2+2iq_1)(1-\chi_1^2))\operatorname{op}(\chi_+)v \\ & - [D_{x_n}, \operatorname{op}(\rho_1\chi_1)]u + \operatorname{op}(R_1)u. \end{aligned} \tag{22}$$

By (19), (20) and (22), using that $(1-\chi_1)\chi_+ = 0$, $(1-\chi_1^2)\chi_+ = 0$ and $\|u\|_{H^1} \le C\|v\|_{H^1}$, we obtain by symbolic calculus

$$\|f_2\|_{L^2(x_n>0)} \le C\|P_\varphi v\|_{L^2(x_n>0)} + Ch\|v\|_{H^1(x_n>0)}. \tag{23}$$

2.2.1 Estimate of $(D_{x_n} - \operatorname{op}(\rho_1\chi_1))u$

Denoting $z = (D_{x_n} - \operatorname{op}(\rho_1\chi_1))u \in L^2(x_n > 0)$ (since $g \in H^1(\mathbb{R}^n)$), we have by (21)

$$(D_{x_n} - \operatorname{op}(\rho_2\chi_1))\underline{z} = \underline{f_2} - ihz_{|x_n=0}\delta_{x_n=0}. \tag{24}$$

Let $\chi \in S^0$ such that $\chi = 1$ if $|\xi|$ is large, $\chi = 1$ in a neighborhood of $\operatorname{supp}\chi_+ \times \mathbb{R}_{\xi_n}$ and

$$\operatorname{supp}\chi \cap \{(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n,\ \xi_n - \rho_2(x,\xi') = 0\} = \emptyset.$$

This is indeed possible. If $|\xi'|$ is large enough then $q_2(x,\xi') \ge C|\xi'|^2$ and $\operatorname{Im}\rho_2 < -C|\xi'|$, $\mu(x,\xi') \ge 0$ and in this region $\xi_n - \rho_2(x,\xi') \ne 0$. If $|\xi'|$ is bounded, $\rho_2(x,\xi')$ is also bounded and if $|\xi_n|$ large, $\xi_n - \rho_2(x,\xi') \ne 0$. Now, if $|\xi|$ is bounded then, on the support of χ_+, $\operatorname{Im}\rho_2 < -\partial_{x_n}\varphi(x)$ by Lemma B.1 and $\xi_n - \rho_2(x,\xi') \ne 0$.
Moreover, by the same arguments, we obtain that $|\xi_n - \rho_2\chi_1| \ge c\langle\xi\rangle$ on $\operatorname{supp}\chi$.

Observe now that $\xi_n - \rho_2\chi_1 \in S\left(\langle\xi\rangle, dx^2 + \frac{d\xi'^2}{\langle\xi'\rangle^2} + \frac{d\xi_n^2}{\langle\xi\rangle^2}\right)$. The metric $\tilde{g} = dx^2 + \frac{d\xi'^2}{\langle\xi'\rangle^2} + \frac{d\xi_n^2}{\langle\xi\rangle^2}$ is slowly varying, semi-classical σ-temperate, the weights $\langle\xi\rangle$, $\langle\xi'\rangle$ are $\tilde{g}$-continuous and semi-classical $\sigma, \tilde{g}$-temperate (see definitions in Appendix A and Lemma A.1 with $\varepsilon = 1$ with a change of variables and dimension).
We set

$$q(x,\xi) = \frac{\chi(x,\xi)}{\xi_n - (\rho_2\chi_1)(x,\xi')} \in S\left(\langle\xi\rangle^{-1}, dx^2 + \frac{d\xi'^2}{\langle\xi'\rangle^2} + \frac{d\xi_n^2}{\langle\xi\rangle^2}\right).$$

We have, by symbolic calculus,

$$\operatorname{Op}(q)\operatorname{Op}(\xi_n - \rho_2\chi_1) = \chi + R$$

where $R \in hS\left(\langle\xi'\rangle^{-1}, dx^2 + \frac{d\xi'^2}{\langle\xi'\rangle^2} + \frac{d\xi_n^2}{\langle\xi\rangle^2}\right)$.
Moreover, we can improve this by $R \in hS\left(\langle\xi\rangle^{-1}, dx^2 + \frac{d\xi'^2}{\langle\xi'\rangle^2} + \frac{d\xi_n^2}{\langle\xi\rangle^2}\right)$. Indeed, the symbol of $\operatorname{Op}(q)\operatorname{Op}(\xi_n)$ is $q\xi_n$ and, using that $\rho_2\chi_1 \in S\left(\langle\xi'\rangle, dx^2 + \frac{d\xi'^2}{\langle\xi'\rangle^2} + \frac{d\xi_n^2}{\langle\xi\rangle^2}\right)$, the symbol of $\operatorname{Op}(q)\operatorname{Op}(-\rho_2\chi_1)$ is $-q\rho_2\chi_1 + R$ where $R \in hS\left(\langle\xi\rangle^{-1}, dx^2 + \frac{d\xi'^2}{\langle\xi'\rangle^2} + \frac{d\xi_n^2}{\langle\xi\rangle^2}\right)$.

Applying $\mathrm{Op}(q)$ in Formula (24), we obtain

$$\underline{z} = \mathrm{Op}(q)\underline{f_2} - ih\,\mathrm{Op}(q)(z_{|x_n=0}\delta_{x_n=0}) + \mathrm{Op}(1-\chi)\underline{z} - \mathrm{Op}(R)\underline{z}. \tag{25}$$

In the sequel, we estimate each terms in the previous equality.
First, we have

$$\begin{aligned}
&\|\,\mathrm{Op}(R)\underline{z}\|_{H^1(x_n>0)} \le \|\,\mathrm{Op}(R)\underline{z}\|_1 \le Ch\|\underline{z}\| \le Ch\|z\|_{L^2(x_n>0)},\\
&\|\,\mathrm{Op}(q)\underline{f_2}\|_{H^1(x_n>0)} \le C\|\,\mathrm{Op}(q)\underline{f_2}\|_1 \le C\|f_2\|_{L^2(x_n>0)}.
\end{aligned} \tag{26}$$

Moreover, using Lemma 2.1, we have

$$z = [D_{x_n} - \mathrm{op}(\rho_1\chi_1)]\,\mathrm{op}(\chi_+)v = \mathrm{op}(\chi_+)[D_{x_n} - \mathrm{op}(\rho_1\chi_1)]v + h\,\mathrm{op}(R_0)v \tag{27}$$

where $R_0 \in S^0_{\mathcal{T}}$.
Let $y = [D_{x_n} - \mathrm{op}(\rho_1\chi_1)]v \in L^2(x_n > 0)$.
We have, by (27), $\underline{z} = \mathrm{op}(\chi_+)\underline{y} + h\underline{\mathrm{op}(R_0)v}$. Thus, we obtain

$$\mathrm{Op}(1-\chi)\underline{z} = \mathrm{Op}(1-\chi)\,\mathrm{op}(\chi_+)\underline{y} + h\,\mathrm{Op}(1-\chi)\underline{\mathrm{op}(R_0)v}. \tag{28}$$

Moreover, since $\mathrm{supp}(1-\chi)\cap\mathrm{supp}(\chi_+) = \emptyset$, one can apply [19, Lemma 2.2] and get that

$$\mathrm{Op}(1-\chi)\,\mathrm{op}(\chi_+) \in \bigcap_{N\in\mathbb{N}} h^N\Psi^{-N}$$

and, consequently,

$$\|\,\mathrm{Op}(1-\chi)\,\mathrm{op}(\chi_+)\underline{y}\|_1 \le h\|y\|_{L^2(x_n>0)} \le Ch\|v\|_{H^1(x_n>0)}. \tag{29}$$

We remark that $D_{x_n}\,\mathrm{Op}(1-\chi) \in S^0$ because $\chi = 1$ when $|\xi|$ large enough. Since R_0 is a tangential symbol, we get

$$\begin{aligned}
\|D_{x_n}\,\mathrm{Op}(1-\chi)\underline{\mathrm{op}(R_0)v}\|_{L^2(x_n>0)} &\le \|D_{x_n}\,\mathrm{Op}(1-\chi)\underline{\mathrm{op}(R_0)v}\|\\
&\le C\|\underline{\mathrm{op}(R_0)v}\| \le C\|v\|_{L^2(x_n>0)}.
\end{aligned} \tag{30}$$

Following (28), (29) and (30), we have

$$\|\,\mathrm{Op}(1-\chi)\underline{z}\|_{H^1(x_n>0)} \le \|\,\mathrm{Op}(1-\chi)\underline{z}\|_1 \le Ch\|v\|_{H^1(x_n>0)}. \tag{31}$$

We have also

$$\begin{aligned}
-ih\,\mathrm{Op}(q)(z_{|x_n=0}\delta_{x_n=0}) &= \frac{1}{(2\pi h)^{n-1}}\int e^{ix'\xi'/h}\left(\frac{1}{2i\pi}\int e^{ix_n\xi_n/h}q(x,\xi',\xi_n)d\xi_n,\right)\hat{z}_{|x_n=0}(\xi')d\xi'\\
&= \mathrm{op}(t)(z_{|x_n=0})
\end{aligned} \tag{32}$$

where $\hat{z}_{|x_n=0}(\xi')$ is the Fourier transform with respect to x' taken at $x_n = 0$, the formula should be understood as an oscillating integral and we have set

$$t(x,\xi') = \frac{1}{2i\pi}\int e^{ix_n\xi_n/h}q(x,\xi',\xi_n)d\xi_n.$$

If one also requires $\chi = 1$ for $|\xi_n| \ge C\langle\xi'\rangle$ and $\xi_n \in \mathbb{C}$ (which is compatible with the definition of χ on $\mathbb{R}$), we get

$$t(x,\xi') = \frac{1}{2i\pi}\int_\gamma e^{ix_n\xi_n/h}\frac{\chi(x,\xi',\xi_n)}{\xi_n - (\rho_2\chi_1)(x,\xi')}d\xi_n \tag{33}$$

where we integrate on the new contour $\gamma = [-C\langle\xi'\rangle, C\langle\xi'\rangle] \cup \{\xi_n \in \mathbb{C}; |\xi_n| = C\langle\xi'\rangle, \mathrm{Im}(\xi_n) > 0\}$.
By (33), we obtain that, for all $l \in \mathbb{N}$, all $\alpha,\ \beta \in \mathbb{N}^{n-1}$, there exists $C > 0$ such that

$$|\partial_{x_n}^l \partial_{x'}^\alpha \partial_{\xi'}^\beta t(x,\xi')| \leq C h^{-l} \langle\xi'\rangle^{l-|\beta|}. \tag{34}$$

Let now $\chi_2(x',\xi') \in \mathscr{C}^\infty(\mathbb{R}^{n-1}\times\mathbb{R}^{n-1})$ a cut-off function such that $(\chi_+)_{|x_n=0}$ is supported in the interior of $\{\chi_2 = 1\}$ and $(\chi_1)_{|x_n=0} = 1$ on a neighborhood of the support of χ_2.
One may write

$$\mathrm{op}(t)(z_{|x_n=0}) = \mathrm{op}(t\chi_2)(z_{|x_n=0}) + \mathrm{op}((1-\chi_2)t)(z_{|x_n=0}). \tag{35}$$

First, we get

$$\begin{aligned}\mathrm{op}((1-\chi_2)t)(z_{|x_n=0}) &= \mathrm{op}((1-\chi_2)t)\,[(D_{x_n} - \mathrm{op}(\rho_1\chi_1)\,\mathrm{op}(\chi_+))\,v]_{|x_n=0}\\ &= \mathrm{op}((1-\chi_2)t)\,\mathrm{op}((\chi_+)_{|x_n=0})\,((D_{x_n} - \mathrm{op}(\rho_1\chi_1))v)_{|x_n=0}\\ &\quad + \mathrm{op}((1-\chi_2)t)\,([D_{x_n} - \mathrm{op}(\rho_1\chi_1), \mathrm{op}(\chi_+)]\,v)_{|x_n=0}\,. \end{aligned}\tag{36}$$

By symbolic calculus and as $\mathrm{supp}(1-\chi_2)\cap\mathrm{supp}\,\chi_+ = \emptyset$, the asymptotic expansion of the symbols of $\mathrm{op}((1-\chi_2)t)\,\mathrm{op}((\chi_+)_{|x_n=0})$ and $\mathrm{op}((1-\chi_2)t)\,[D_{x_n} - \mathrm{op}(\rho_1\chi_1), \mathrm{op}(\chi_+)]_{|x_n=0}$ are null (taking account that $[D_{x_n} - \mathrm{op}(\rho_1\chi_1), \mathrm{op}(\chi_+)]$ is a tangential operator). Hence by trace formula, we have

$$\|\,\mathrm{op}(\langle\xi'\rangle)\,\mathrm{op}((1-\chi_2)t)(z_{|x_n=0})\|_{L^2(x_n>0)} \leq Ch\|v\|_{H^1(x_n>0)} + Ch|D_{x_n}v_{|x_n=0}|_{-1/2}. \tag{37}$$

On the support of χ_2 we have $\chi_{|x_n=0} = 0$ and, following (33), we deduce

$$(t\chi_2)(x,\xi') = \frac{1}{2i\pi}\chi_2(x,\xi')\int_\gamma e^{ix_n\xi_n/h}\frac{1}{\xi_n - \rho_2(x,\xi')}d\xi_n = 0 \tag{38}$$

by residue formula and since, by Lemma B.1, $\mathrm{Im}\,\rho_2 < 0$ on the support of χ_2.

To estimate the L^2 norm of $\partial_{x_n}\mathrm{op}(t)(z_{|x_n=0}) = \mathrm{op}(\partial_{x_n}t)(z_{|x_n=0})$, we proceed in the same way. Actually, $\partial_{x_n}t \in h^{-1}S^1_{\mathcal{T}}$ and we have to use (36).
By the same support argument used to obtain (37), we get

$$\|\partial_{x_n}\mathrm{op}((1-\chi_2)t)(z_{|x_n=0})\|_{L^2(x_n>0)} \leq Ch\|v\|_{H^1(x_n>0)} + Ch|D_{x_n}v_{|x_n=0}|_{-1/2}. \tag{39}$$

Analogously, the equation (38) become

$$(\partial_{x_n}t\chi_2)(x,\xi') = \frac{h^{-1}}{2\pi}\chi_2(x,\xi')\int_\gamma e^{ix_n\xi_n/h}\frac{\xi_n}{\xi_n - \rho_2(x,\xi')}d\xi_n = 0. \tag{40}$$

Following (35), (37), (38), (39) and (40), we deduce

$$\|\,\mathrm{op}(t)(z_{|x_n=0})\|_{H^1(x_n>0)} \leq Ch\|v\|_{H^1(x_n>0)} + Ch|D_{x_n}v_{|x_n=0}|_{-1/2}. \tag{41}$$

Finally, using (25), (26), (31), (32), (41), and for all h small enough, we obtain

$$\begin{aligned}\|\,z\|_{H^1(x_n>0)} &\leq C\|f_2\|_{L^2(x_n>0)} + Ch\|v\|_{H^1(x_n>0)} + Ch|(D_{x_n}v)_{|x_n=0}|_{-1/2}\\ &\leq C\|P_\varphi v\|_{L^2(x_n>0)} + Ch\|v\|_{H^1(x_n>0)} + Ch|(D_{x_n}v)_{|x_n=0}|_{-1/2}\end{aligned}\tag{42}$$

where we have used (23).

2.2.2 Estimates of v_0 and v_1

The goal is now to find an equation on v_0 and v_1 (see their definitions in (16)) . We remind that $z = [D_{x_n} - \mathrm{op}(\rho_1\chi_1)]u$ and, since $u = \mathrm{op}(\chi_+)v$, we have then

$$\begin{aligned}&(D_{x_n}v)_{|x_n=0} - (\mathrm{op}(\rho_1\chi_1)v)_{|x_n=0} = f_3\\ &\text{where } f_3 = z_{|x_n=0} + (1-\mathrm{op}(\chi_+))(D_{x_n}v)_{|x_n=0} - \mathrm{op}(\rho_1\chi_1)(1-\mathrm{op}(\chi_+))v_{|x_n=0}.\end{aligned}\tag{43}$$

Following (17), we have

$$\begin{aligned}&v_1 - \mathrm{op}((\rho_1 + i(\partial_{x_n}\varphi)_{|x_n=0})\chi_1)v_0 = f_4\\ &\text{where } f_4 = f_3 - G_1 + \mathrm{op}((\rho_1\chi_1)_{|x_n=0})G_0 + \mathrm{op}(i(\partial_{x_n}\varphi)_{|x_n=0}(1-\chi_{1|x_n=0}))v_0.\end{aligned}\tag{44}$$

Remark 4. We may now explain the main difficulty faced to solve this equation. Coming back to our model case detailled in the beginning of section 2, one has

$$\rho_1 + i(\partial_{x_n}\varphi)_{|x_n=0} = i|\xi'|,$$

so that equation (44) takes the form, up to some remainder term and where f is some data,

$$v_1 - i\,\mathrm{op}(|\xi'|)v_0 = f.$$

Since the symbol $|\xi'|$ does not satisfy the transmission condition, one cannot use the usual algebra of pseudodifferential operators. We will overcome this problem writing a factorization of the form

$$|\xi'| = (\xi_1 + i|\xi''|)^{1/2}(\xi_1 - i|\xi''|)^{1/2}$$

and using that the operators of symbols $(\xi_1 \pm i|\xi''|)^{1/2}$ preserves functions with support in $\{\mp x_1 > 0\}$.

Coming back to our remainder estimates, one has

$$|D_{x_n}v_{|x_n=0}|_{-1/2} \leq C|v_0|_{1/2} + C|v_1|_{-1/2} + C|G_1|_{-1/2}$$

and, since $1-\chi_{1|x_n=0} = (1-\chi_{1|x_n=0})(1-\chi_+)$,

$$\mathrm{op}(1-\chi_{1|x_n=0}) - \mathrm{op}((1-\chi_{1|x_n=0}))\,\mathrm{op}(1-\chi_+) \in h\Psi^0_{\mathcal{T}} \tag{45}$$

which gives

$$|\,\mathrm{op}\,(1-\chi_{1|x_n=0})v_0| \leq C|\,\mathrm{op}\,(1-\chi_+)v_0| + Ch|v_0|.$$

Consequently, by (42), (43), (44) and trace formula, we obtain

$$\begin{aligned} h^{1/2}|f_4|_{-1/2} \leq Ch^{1/2}|G_1|_{-1/2} + Ch^{1/2}|G_0|_{1/2} + C\|P_\varphi v\|_{L^2(x_n>0)} + Ch\|v\|_{H^1(x_n>0)} + Ch|v_1|_{-1/2} \\ + Ch|v_0| + Ch^{1/2}|\,\mathrm{op}\,(1-\chi_+)D_{x_n}v_{|x_n=0}|_{-1/2} + Ch^{1/2}|\,\mathrm{op}\,(1-\chi_+)v_{|x_n=0}|_{1/2}. \end{aligned} \tag{46}$$

Let now

$$\lambda_-^s(\xi') = \left(\xi_1 + i\sqrt{|\xi''|^2 + (\varepsilon\partial_{x_n}\varphi(0))^2}\right)^s,$$
$$\lambda_+^s(\xi') = \left(\xi_1 - i\sqrt{|\xi''|^2 + (\varepsilon\partial_{x_n}\varphi(0))^2}\right)^s.$$

Since λ_-^s is holomorphic function in $\operatorname{Im}\xi_1 > 0$ and using an adapted version of the Paley-Wiener theorem (see Theorem 7.4.3 in [14]), one gets that λ_-^s is the Fourier transform of a distribution supported on $\{x_1 \leq 0\}$ and, for analogous reason, λ_+^s is the Fourier transform of a distribution supported on $\{x_1 \geq 0\}$. This justifies the indices $+$ and $-$.

We set $z_1 = \mathrm{op}(\lambda_-^{-1/2})v_1 \in L^2(x_n=0)$ and $z_0 = \mathrm{op}(\lambda_+^{1/2})v_0 \in L^2(x_n=0)$. We have $\operatorname{supp} z_1 \subset \{x_1 \leq 0\}$ and $\operatorname{supp} z_0 \subset \{x_1 \geq 0\}$. Moreover, by (44),

$$z_1 - \mathrm{op}(\lambda_-^{-1/2})\,\mathrm{op}(((\rho_1 + i\partial_{x_n}\varphi)\chi_1)_{|x_n=0})\,\mathrm{op}(\lambda_+^{-1/2})z_0 = f_5$$

where

$$h^{1/2}|f_5| \leq C_\varepsilon h^{1/2}|f_4|_{-1/2}. \tag{47}$$

As z_1 supported in $\{x_1 \leq 0\}$, we have

$$r_{x_1>0}\,\mathrm{op}(\lambda_-^{-1/2})\,\mathrm{op}(((\rho_1 + i\partial_{x_n}\varphi)\chi_1)_{|x_n=0})\,\mathrm{op}(\lambda_+^{-1/2})z_0 = -r_{x_1>0}f_5 \tag{48}$$

where we denote here $r_{x_1>0}z := z_{|x_1>0}$ the restriction of z to $\{x_1 > 0\}$.
Following the notations introduced in Appendix A, we note that $\lambda_\pm^{-1/2} \in S(\langle\xi\rangle_\varepsilon^{-1/2}, g)$.
Moreover, one has $(\rho_1 + i(\partial_{x_n}\varphi)_{|x_n=0})\chi_1 \in S(\langle\xi\rangle_\varepsilon, g)$. Consequently, by symbolic calculus, we have

$$\mathrm{op}(\lambda_-^{-1/2})\,\mathrm{op}(((\rho_1 + i\partial_{x_n}\varphi)\chi_1)_{|x_n=0})\,\mathrm{op}(\lambda_+^{-1/2}) = \mathrm{op}(a) + h\,\mathrm{op}(b) \tag{49}$$

where

$$\begin{cases} a = \lambda_-^{-1/2}((\rho_1 + i\partial_{x_n}\varphi)\chi_1)_{|x_n=0}\lambda_+^{-1/2}, \\ b \in S(\langle\xi'\rangle\langle\xi'\rangle_\varepsilon^{-1}\langle\xi''\rangle_\varepsilon^{-1}, g). \end{cases}$$

Remark 5. To guide the reader, we shall also explain what happens at this milestone in our model case. As explained above, one has

$$(\rho_1 + i\partial_{x_n}\varphi) = i|\xi'|$$

and consequently, in some sense precised below,

$$a = i + O(\varepsilon).$$

The very simple form of this operator explain that one should now get the estimates desired on z_0 and then on v_0 and v_1.

The estimates on b implies that $|b(x',\xi')| \leq C/\varepsilon^2$ (because $\langle\xi'\rangle\langle\xi'\rangle_\varepsilon^{-1}\langle\xi''\rangle_\varepsilon^{-1} \leq \varepsilon^{-2}$) and $b \in S(1,g)$ with semi-norms depending on ε. We can apply the Lemma A.3 to get

$$\forall\varepsilon > 0,\ \exists C_\varepsilon > 0;\ |\operatorname{op}(b)z_0|_{L^2(x_1>0)} \leq |\operatorname{op}(b)z_0| \leq C_\varepsilon|z_0|. \tag{50}$$

Second, we have $a = a_{|x'=0} + c$ and $a \in S(\langle\xi'\rangle\langle\xi'\rangle_\varepsilon^{-1}, g)$ which imply that $a \in S(1,g)$ with semi-norms depending on ε and $|\partial_{x'}c(x',\xi')| \leq C\varepsilon^{-1}$ (because $\langle\xi'\rangle\langle\xi'\rangle_\varepsilon^{-1} \leq \varepsilon^{-1}$).
We can apply Lemma A.4 to c and obtain, if the radius satisfies $\kappa \leq \varepsilon^2$ (see the hypotheses of Theorem 2) and since v_0 is supported in B_κ,

$$|\operatorname{op}(c)z_0|_{L^2(x_1>0)} \leq C\varepsilon|z_0| + C_\varepsilon h^{1/2}|v_0|_{1/2}. \tag{51}$$

Next, we use the assumption $|\partial_{x'}\varphi(0)| \leq \varepsilon\partial_{x_n}\varphi(0)$, the change of variables found in Lemma B.2 and Lemma B.1.
Writing $V = \frac{\partial_{x'}\varphi(0)}{\partial_{x_n}\varphi(0)}$ and $\eta' = \frac{\xi'}{\partial_{x_n}\varphi(0)}$, we have by (97)

$$\rho_1(0,\xi') + i\partial_{x_n}\varphi(0) = i\partial_{x_n}\varphi(0)\rho(\eta')$$

with

$$\rho(\eta') = \sqrt{|\eta'|^2 - |V|^2 + 2i\eta'.V}\ .$$

One may write

$$a_{|x'=0}(\xi') = i\frac{\partial_{x_n}\varphi(0)\ \rho(\eta')\ (\chi_1)_{|x=0}}{\sqrt{|\xi'|^2 + (\varepsilon\partial_{x_n}\varphi(0))^2}} = i(\chi_1)(0,\xi') + id(\xi') \tag{52}$$

where $d(\xi') = \tilde{d}(\eta') \times \chi_1(0,\xi')$ and

$$\tilde{d}(\eta') = \frac{\rho(\eta')}{\sqrt{|\eta'|^2 + \varepsilon^2}} - 1 = \frac{\rho(\eta') - \sqrt{|\eta'|^2 + \varepsilon^2}}{\sqrt{|\eta'|^2 + \varepsilon^2}}.$$

We remark that

$$\forall\eta',\ |\eta'| \leq \sqrt{|\eta'|^2 + \varepsilon^2} \leq |\eta'| + \varepsilon$$

and, using now (98), we thus obtain, since $|V| \leq \varepsilon$ and for ε sufficiently small,

$$|\tilde{d}(\eta')| \leq C\varepsilon$$

since, by (96), $|\eta'| \geq \delta$ on $\operatorname{supp}(\chi_1) \subset \mathcal{F}_\alpha$.
Finally, one has

$$|d(\xi')| \leq C\varepsilon \text{ and } d(\xi') \in S(1,g)$$

and we can apply Lemma A.3 to deduce

$$|\operatorname{op}(d)z_0|_{L^2(x_1>0)} \leq |\operatorname{op}(d)z|_0 \leq (C\varepsilon + C_\varepsilon h^{1/2})|z_0|. \tag{53}$$

Following (48), (49) and (52), we obtain

$$ir_{x_1>0}z_0 + r_{x_1>0}\operatorname{op}(hb + c + id(\chi_1)_{|x_n=0} + (1 - (\chi_1)_{|x_n=0}))z_0 = -r_{x_1>0}f_5. \tag{54}$$

On the other hand, using (45) again, we have

$$\begin{aligned}|\operatorname{op}(1-(\chi_1)_{|x_n=0})z_0| &\leq C|\operatorname{op}(1-\chi_+)z_0|+Ch|z_0|\\ &\leq C|\operatorname{op}(1-\chi_+)v_0|_{1/2}+Ch|v_0|_{1/2}\end{aligned}$$

and, following (50), (51), (53) and (54), we deduce

$$\begin{aligned}|z_0|_{L^2(x_1>0)} &\leq (C\varepsilon+C_\varepsilon h^{1/2})|z_0|+C_\varepsilon h^{1/2}|v_0|_{1/2}+C|\operatorname{op}(1-\chi_+)v_0|_{1/2}+|f_5|\\ &\leq C\varepsilon|z_0|+C_\varepsilon h^{1/2}|v_0|_{1/2}+C|\operatorname{op}(1-\chi_+)v_0|_{1/2}+|f_5|.\end{aligned}$$

It is clear that $|z_0|_{L^2(x_1>0)}=|z_0|$. Taking ε small enough, we have then

$$\forall h\in(0,h_0],\ |z_0|\leq Ch^{1/2}|v_0|_{1/2}+C|\operatorname{op}(1-\chi_+)v_0|_{1/2}+C|f_5|.$$

Using $z_0=\operatorname{op}(\lambda_+^{1/2})v_0$ and for h_0 small enough, we deduce

$$|v_0|_{1/2}\leq C|\operatorname{op}(1-\chi_+)v_0|_{1/2}+C|f_5|.$$

Following (46) and (47) we have, using (17) and for h_0 small enough,

$$\begin{aligned}h^{1/2}|v_0|_{1/2}\leq Ch^{1/2}|G_1|_{-1/2}+Ch^{1/2}|G_0|_{1/2}+C\|P_\varphi v\|_{L^2(x_n>0)}+Ch\|v\|_{H^1(x_n>0)}+Ch|v_1|_{-1/2}\\ +Ch^{1/2}|\operatorname{op}(1-\chi_+)D_{x_n}v_{|x_n=0}|_{-1/2}+Ch^{1/2}|\operatorname{op}(1-\chi_+)v_{|x_n=0}|_{1/2}.\end{aligned}\tag{55}$$

Using (55) in (44) and by (46), we obtain also, for h_0 small enough,

$$\begin{aligned}h^{1/2}|v_1|_{-1/2}\leq Ch^{1/2}|G_1|_{-1/2}+Ch^{1/2}|G_0|_{1/2}+C\|P_\varphi v\|_{L^2(x_n>0)}+Ch\|v\|_{H^1(x_n>0)}+Ch|v_1|_{-1/2}\\ +Ch^{1/2}|\operatorname{op}(1-\chi_+)D_{x_n}v_{|x_n=0}|_{-1/2}+Ch^{1/2}|\operatorname{op}(1-\chi_+)v_{|x_n=0}|_{1/2}.\end{aligned}$$

One now remarks that the term $|v_1|$ on the right-hand side of this inequality can be absorbed if h_0 is sufficiently small. Recalling now (17) and (18), one gets, provided h_0 is small enough,

$$\begin{aligned}h^{1/2}|v_{|x_n=0}|_{1/2}\leq Ch^{3/2}|e^{\varphi/h}g_1|_{-1/2}+Ch^{1/2}|e^{\varphi/h}g_0|_{1/2}+C\|P_\varphi v\|_{L^2(x_n>0)}+Ch\|v\|_{H^1(x_n>0)}\\ +Ch^{1/2}|\operatorname{op}(1-\chi_+)D_{x_n}v_{|x_n=0}|_{-1/2}+Ch^{1/2}|\operatorname{op}(1-\chi_+)v_{|x_n=0}|_{1/2}\end{aligned}\tag{56}$$

and

$$\begin{aligned}h^{1/2}|D_{x_n}v_{|x_n=0}|_{-1/2}\leq Ch^{3/2}|e^{\varphi/h}g_1|_{-1/2}+Ch^{1/2}|e^{\varphi/h}g_0|_{1/2}+C\|P_\varphi v\|_{L^2(x_n>0)}+Ch\|v\|_{H^1(x_n>0)}\\ +Ch^{1/2}|\operatorname{op}(1-\chi_+)D_{x_n}v_{|x_n=0}|_{-1/2}+Ch^{1/2}|\operatorname{op}(1-\chi_+)v_{|x_n=0}|_{1/2}.\end{aligned}\tag{57}$$

2.2.3 Estimate of u in $H^1(x_n>0)$

To estimate the H^1-norm of u we use the Carleman technique.
First, we have

$$\begin{aligned}\|(D_{x_n}-\operatorname{op}(\rho_1\chi_1))u\|^2_{L^2(x_n>0)} &= \|(D_{x_n}-\operatorname{op}(\operatorname{Re}\rho_1\chi_1))u\|^2_{L^2(x_n>0)}+\|\operatorname{op}(\operatorname{Im}\rho_1\chi_1)u\|^2_{L^2(x_n>0)}\\ &+ 2\operatorname{Re}((D_{x_n}-\operatorname{op}(\operatorname{Re}\rho_1\chi_1))u|-i\operatorname{op}(\operatorname{Im}\rho_1\chi_1)u)_{L^2(x_n>0)}\\ &= \|(D_{x_n}-\operatorname{op}(\operatorname{Re}\rho_1\chi_1))u\|^2_{L^2(x_n>0)}+\|\operatorname{op}(\operatorname{Im}\rho_1\chi_1)u\|^2_{L^2(x_n>0)}\\ &+ \operatorname{Re}(i[\operatorname{op}(\operatorname{Im}\rho_1\chi_1),D_{x_n}-\operatorname{op}(\operatorname{Re}\rho_1\chi_1)]u|u)\\ &- h\operatorname{Re}(\operatorname{op}(\operatorname{Im}\rho_1\chi_1)u|u)_0\\ &+ h\left(\left(L_0(D_{x_n}-\operatorname{op}(\operatorname{Re}\rho_1\chi_1))+L_1\operatorname{op}(\operatorname{Im}\rho_1\chi_1)\right)u|u\right),\end{aligned}\tag{58}$$

where $(.,.)$ (resp. $(.,.)_0$) denotes the standard scalar product on $\{x_n>0\}$ (resp. $\{x_n=0\}$) and L_j are tangential operators of orders 0.
Moreover, one has

$$|(\operatorname{op}(\operatorname{Im}\rho_1\chi_1)u|u)_0|\leq C|u_{|x_n=0}|^2_{1/2}\tag{59}$$

and, for any $C' > 0$,

$$h\left|\big((L_0(D_{x_n} - \operatorname{op}(\operatorname{Re}\rho_1\chi_1)) + L_1 \operatorname{op}(\operatorname{Im}\rho_1\chi_1)\big)u|u)\right| \\ \leq \frac{1}{C'}\big(\|(D_{x_n} - \operatorname{op}(\operatorname{Re}\rho_1\chi_1))u\|^2_{L^2(x_n>0)} + \|\operatorname{op}(\operatorname{Im}\rho_1\chi_1)u\|^2_{L^2(x_n>0)}\big) + Ch^2\|u\|^2_{L^2(x_n>0)}. \quad (60)$$

The commutator term $i[\operatorname{op}(\operatorname{Im}\rho_1\chi_1), D_{x_n} - \operatorname{op}(\operatorname{Re}\rho_1\chi_1)]$ is a tangential 1-order operator and has $h\{\operatorname{Im}\rho_1\chi_1, \xi_n - \operatorname{Re}\rho_1\chi_1\}$ for principal symbol.
Taking account that $p_\varphi = (\xi_n - \rho_1)(\xi_n - \rho_2)$, we have

$$\begin{aligned}\{\operatorname{Re} p_\varphi, \operatorname{Im} p_\varphi\} &= \frac{1}{2i}\{\bar{p}_\varphi, p_\varphi\} \\ &= |\xi_n - \rho_2|^2\{\operatorname{Im}\rho_1, \xi_n - \operatorname{Re}(\rho_1)\} + O(\xi_n - \rho_1) + O(\xi_n - \bar{\rho}_1).\end{aligned}$$

Hence by hypothesis (4) we obtain that

$$\operatorname{Im}\rho_1 = 0,\ \xi_n = \operatorname{Re}\rho_1 \ \Rightarrow\ \{\operatorname{Im}\rho_1, \xi_n - \operatorname{Re}(\rho_1)\} > 0.$$

Furthermore, noting that $\{\operatorname{Im}\rho_1, \xi_n - \operatorname{Re}(\rho_1)\}$ is in fact independent of ξ_n, we have

$$\operatorname{Im}\rho_1 = 0 \ \Rightarrow\ \{\operatorname{Im}\rho_1, \xi_n - \operatorname{Re}(\rho_1)\} > 0$$

and this classically implies that there exists $C_1 > 0$ and $C_2 > 0$ such that, on the support of χ_1,

$$\{\operatorname{Im}\rho_1, \xi_n - \operatorname{Re}(\rho_1)\} + C_1\langle\xi'\rangle^{-1}|\operatorname{Im}\rho_1|^2 \geq C_2\langle\xi'\rangle.$$

Using the standard Gårding inequality (see e.g. [24, Theorem 3.5.8]), we deduce

$$\begin{aligned} & C_2\|\operatorname{op}(\langle\xi'\rangle^{1/2})\operatorname{op}(\chi_1)u\|^2_{L^2(x_n>0)} - Ch\|u\|^2_{L^2(x_n>0)} \quad (61) \\ \leq\ & \operatorname{Re}(i[\operatorname{op}(\operatorname{Im}\rho_1\chi_1), D_{x_n} - \operatorname{op}(\operatorname{Re}\rho_1\chi_1)]u|u) + C_1\|\operatorname{op}(\operatorname{Im}\rho_1\chi_1)u\|^2_{L^2(x_n>0)}.\end{aligned}$$

On the other hand, as $\chi_1 = 1$ on the support of χ_+, we have $\operatorname{op}(\chi_+) = \operatorname{op}(\chi_1)\operatorname{op}(\chi_+) + h\operatorname{op}(r_{-1})$ where $r_{-1} \in S^{-1}_{\mathcal{T}}$. Thus

$$\|\operatorname{op}(\langle\xi'\rangle^{1/2})u\|^2_{L^2(x_n>0)} \leq C\|\operatorname{op}(\langle\xi'\rangle^{1/2})\operatorname{op}(\chi_1)u\|^2_{L^2(x_n>0)} + Ch\|v\|^2_{L^2(x_n>0)}. \quad (62)$$

From (58), (59), (60), (61) and (62), we get

$$\begin{aligned} & \tfrac{1}{2}\|\operatorname{op}(\operatorname{Im}\rho_1\chi_1)u\|^2_{L^2(x_n>0)} + C_2\|u\|^2_{H^{1/2}(x_n>0)} \quad (63) \\ \leq\ & \|(D_{x_n} - \operatorname{op}(\rho_1\chi_1))u\|^2_{L^2(x_n>0)} + Ch|u_{|x_n=0}|^2_{1/2} + Ch\|v\|^2_{L^2(x_n>0)}\end{aligned}$$

for C' chosen sufficiently large.
Let now $C > 0$ given by Lemma B.1 such that $\operatorname{Im}\rho_1(x,\xi') \geq \delta|\xi'|$ for $|\xi'| \geq C$. Let also $\chi_H \in \mathscr{C}^\infty(\mathbb{R}^{n-1})$ such that

$$\chi_H(\xi') = \begin{cases} 1 \text{ if } |\xi'| \geq C + 1, \\ 0 \text{ if } |\xi'| \leq C. \end{cases}$$

Note that in particular $\chi_1 = 1$ on the support of χ_H so that

$$\operatorname{op}(\langle\xi'\rangle)\operatorname{op}(\chi_H) - \operatorname{op}(\chi_H\langle\xi'\rangle/(\operatorname{Im}\rho_1))\operatorname{op}((\operatorname{Im}\rho_1)\chi_1) \in h\Psi^0_{\mathcal{T}}.$$

Using symbolic calculus, we consequently obtain

$$\|\operatorname{op}(\langle\xi'\rangle)\operatorname{op}(\chi_H)u\|_{L^2(x_n>0)} \leq C\|\operatorname{op}((\operatorname{Im}\rho_1)\chi_1))u\|_{L^2(x_n>0)} + Ch\|u\|_{L^2(x_n>0)}. \quad (64)$$

If h_0 is small enough, using (63), (64) and that $1 - \chi_H$ is compactly supported, we get

$$\begin{aligned}\|\operatorname{op}(\langle\xi'\rangle)u\|_{L^2(x_n>0)} &\leq \|\operatorname{op}(\langle\xi'\rangle)\operatorname{op}(\chi_H)u\|_{L^2(x_n>0)} + \|\operatorname{op}(\langle\xi'\rangle)(1 - \operatorname{op}(\chi_H))u\|_{L^2(x_n>0)} \\ &\leq C\|(D_{x_n} - \operatorname{op}(\rho_1\chi_1))u\|_{L^2(x_n>0)} + Ch^{1/2}|u_{|x_n=0}|_{1/2} + Ch^{1/2}\|v\|_{L^2(x_n>0)}. \quad (65)\end{aligned}$$

We have also

$$\begin{aligned}\|D_{x_n}u\|_{L^2(x_n>0)} &\le \|(D_{x_n}-\mathrm{op}(\rho_1\chi_1))u\|_{L^2(x_n>0)} + \|\,\mathrm{op}(\rho_1\chi_1)u\|_{L^2(x_n>0)}\\ &\le \|(D_{x_n}-\mathrm{op}(\rho_1\chi_1))u\|_{L^2(x_n>0)} + C\|\,\mathrm{op}(\langle\xi'\rangle)u\|_{L^2(x_n>0)}. \end{aligned}\tag{66}$$

Then, by (65) and (66), one gets

$$\|\,\mathrm{op}(\chi_+)v\|_{H^1(x_n>0)} \le C\|(D_{x_n}-\mathrm{op}(\rho_1\chi_1))u\|_{L^2(x_n>0)} + Ch^{1/2}|u_{|x_n=0}|_{1/2} + Ch^{1/2}\|v\|_{L^2(x_n>0)}.$$

Using now (42), we finally deduce

$$\begin{aligned}&\|\,\mathrm{op}(\chi_+)v\|_{H^1(x_n>0)}\\ &\quad\le C\left(\|P_\varphi v\|_{L^2(x_n>0)} + h^{1/2}\|v\|_{H^1(x_n>0)} + h^{1/2}|v_{|x_n=0}|_{1/2} + h|(D_{x_n}v)_{|x_n=0}|_{-1/2}\right).\end{aligned}\tag{67}$$

2.3 End of the proof

We now collect the results of Sections 2.1 and 2.2.
We first note that is possible to choose α so small that $\chi_- = 1$ on $\mathrm{supp}(1-\chi_+)$.
Doing so, one has $\mathrm{op}\,(1-\chi_+) - \mathrm{op}\,(1-\chi_+)\,\mathrm{op}\,(\chi_-) \in h\Psi^0_{\mathcal{T}}$ which gives

$$\begin{aligned}&|\,\mathrm{op}\,(1-\chi_+)D_{x_n}v_{|x_n=0}|_{-1/2} + |\,\mathrm{op}\,(1-\chi_+)v_{|x_n=0}|_{1/2}\\ &\le C\left(|\,\mathrm{op}\,(\chi_-)D_{x_n}v_{|x_n=0}|_{-1/2} + |\,\mathrm{op}\,(\chi_-)v_{|x_n=0}|_{1/2} + h|D_{x_n}v_{|x_n=0}|_{-1/2} + h|v_{|x_n=0}|_{1/2}\right).\end{aligned}$$

Summing up (15) and (56), (57), we now deduce the final trace estimate

$$\begin{aligned}&|D_{x_n}v_{|x_n=0}|_{-1/2} + |v_{|x_n=0}|_{1/2}\\ &\le C\left(h|e^{\varphi/h}g_1| + |e^{\varphi/h}g_0|_1 + h^{-1/2}\|P_\varphi v\|_{L^2(x_n>0)} + h^{1/2}\|v\|_{H^1(x_n>0)}\right)\end{aligned}\tag{68}$$

provided h_0 is sufficiently small.

Proceeding in the same way, one can deduce that, provided h is sufficiently small,

$$\|v\|_{H^1(x_n>0)} \le C\left(\|\,\mathrm{op}(\chi_-)v\|_{H^1(x_n>0)} + \|\,\mathrm{op}(\chi_+)v\|_{H^1(x_n>0)}\right).$$

Consequently, using (15), (67) and (68), one gets the desired result

$$\|v\|_{H^1(x_n>0)} + |D_{x_n}v_{|x_n=0}|_{-1/2} + |v_{|x_n=0}|_{1/2} \le C\left(|e^{\varphi/h}g_0|_{1/2} + h|e^{\varphi/h}g_1|_{-1/2} + h^{-1/2}\|P_\varphi v\|_{L^2(x_n>0)}\right).$$

3 Proof of Theorem 1

3.1 Different Carleman estimates

We define $X = (-1,1)\times\Omega$ (where $(-1,1)$, diffeomorphic to $\mathbb{R}$, is considered as a manifold without boundary) and we split its boundary into $\partial X_N = (-1,1)\times\partial\Omega_N$ and $\partial X_D = (-1,1)\times\partial\Omega_D$.

Since Ω is a relatively compact smooth open set of $\mathbb{R}^d$, there exists some smooth function f defined in a neighborhood of $\partial\Omega$ such that, near $\partial\Omega$,

$$\begin{cases} y\in\partial\Omega \Leftrightarrow f(y)=0,\\ y\in\Omega \Leftrightarrow f(y)>0.\end{cases}$$

Moreover, near any point $y^*\in\partial\Omega$, one has $df\neq 0$. We may apply Lemma B.2 with $p(y,\xi)=|\xi|^2$, transport y^* to 0 and get that, in new coordinates,

$$\begin{cases} y\in\Omega \Leftrightarrow y_d>0,\\ y\in\partial\Omega \Leftrightarrow y_d=0,\end{cases}\tag{69}$$

with moreover $p(y,\xi)=\xi_d^2 + r(y,\xi')$.
The operator $-\Delta$ consequently takes, in some neighbourhood of 0, the form $-\partial_{y_d}^2 + l(y)\partial_{y_d} - Q\,(y,\partial_{y'})$

with $\partial_{y'} = (\partial_{y_1}, \dots, \partial_{y_{d-1}})$ and Q a smooth elliptic differential operator of order 2.
As in [23], one can find some function $e(y', y_d)$ normalised by $e(y', 0) = 1$ such that the operator $P = -e \circ (\partial_{x_0}^2 + \Delta) \circ 1/e$ takes, in some neighbourhood of 0 and in coordinates $x = (x_0, y)$, the form

$$P = -\partial_{x_d}^2 + R\left(x, \frac{1}{i}\partial_{x'}\right) \tag{70}$$

where $\partial_{x'} = (\partial_{x_0}, \dots, \partial_{x_{d-1}})$, the principal symbol of R is real and satisfy, for some $c > 0$,

$$\forall (x, \xi'),\ \begin{cases} r(x, \xi') \geqslant c|\xi'|^2, \\ r(x, \xi') = |\xi_0|^2 + r_0(x, \xi_0'), \end{cases} \tag{71}$$

with $\xi_0' = (\xi_1, \dots, \xi_{d-1})$.
Moreover, since Γ is a relatively compact smooth submanifold of $\partial\Omega$, there exists some smooth function k defined in a neighborhood of Γ such that, near Γ,

$$\begin{cases} y \in \partial\Omega_D \Leftrightarrow f(y) = 0 \text{ and } k(y) > 0, \\ y \in \partial\Omega_N \Leftrightarrow f(y) = 0 \text{ and } k(y) < 0. \end{cases}$$

Hence, if one works near $y^* \in \Gamma$, one has $dk(y^*) \neq 0$ and we additionally obtain that

$$\begin{cases} y \in \partial\Omega_D \Leftrightarrow y_d = 0 \text{ and } y_1 > 0, \\ y \in \partial\Omega_N \Leftrightarrow y_d = 0 \text{ and } y_1 < 0, \end{cases} \quad \text{that is} \quad \begin{cases} x \in \partial X_D \Leftrightarrow x_d = 0 \text{ and } x_1 > 0, \\ x \in \partial X_N \Leftrightarrow x_d = 0 \text{ and } x_1 < 0, \end{cases} \tag{72}$$

and, along with (70), (71), that

$$r(0, \xi') = |\xi'|^2.$$

On the other hand, if v satisfies (2), then $\tilde{v} = e \times v$ satisfies the following equations

$$\begin{cases} P\tilde{v} = ev_0 & \text{in } X, \\ \tilde{v} = 0 & \text{on } \partial X_D, \\ \partial_\nu \tilde{v} - ia\partial_{x_0}\tilde{v} + b\tilde{v} = v_1 & \text{on } \partial X_N, \end{cases} \tag{73}$$

where, as in [23], we have used the notation $b = -\partial_\nu(1/e)_{|\partial X}$.
To localize and use the local form of our operator, we choose some cut-off function θ with sufficiently small compact support. Let $g = \theta\tilde{v}$ satisfying different problems, depending on the localization chosen.
The first problem is a problem without boundary conditions (if the support of the cut-off is away from ∂X). Moreover, if the support of the cut-off function intersects the boundary, we have three different cases to consider: one where the only boundary condition is of Dirichlet type, one where the only boundary condition is of Neumann type and the last one is a Zaremba boundary problem.

In each situation, we need some adapted Carleman estimates. In the three first situations, these results were obtained by Lebeau and Robbiano: it is Proposition 2 of [22] and Proposition 1, Proposition 2 of [23], recalled below.

As before, we consider $B_\kappa := \{x \in \mathbb{R}^{d+1}; |x| \leq \kappa\}$ and φ a $\mathscr{C}^\infty$ function. We note $\mathscr{C}_0^\infty(B_\kappa)$ the set of $\mathscr{C}^\infty$ functions supported in B_κ.

Proposition 3.1. *Assume that* (3), (4) *hold. Then, there exists $C, h_0 > 0$ such that for any $h \in (0, h_0)$ and for any $g \in \mathscr{C}_0^\infty(B_\kappa)$, the following inequality holds*

$$\|ge^{\varphi/h}\|_{L^2(x_d>0)} + \|h(\partial_x g)e^{\varphi/h}\|_{L^2(x_d>0)} \leq C\left(h^{-1/2}\|h^2P(g)e^{\varphi/h}\|_{L^2(x_d>0)} + |ge^{\varphi/h}| + |h(\partial_{x_d}g)e^{\varphi/h})|)\right).$$

Proposition 3.2. *Assume that* (3), (4) *hold and that*

$$\frac{\partial\varphi}{\partial x_d} > 0 \text{ on } \{x_d = 0\} \cap B_\kappa.$$

Then there exists $C, h_0 > 0$ such that for any $h \in (0, h_0)$, $g_0 \in \mathscr{C}^\infty(\{x_d = 0\})$, and for any $g \in \mathscr{C}_0^\infty(B_\kappa)$ such that

$$g = 0 \text{ on } \{x_d = 0\},$$

the following inequality holds:

$$\|ge^{\varphi/h}\|_{L^2(x_d>0)} + \|h(\partial_x g)e^{\varphi/h}\|_{L^2(x_d>0)} \leq Ch^{-1/2}\|h^2 P(g)e^{\varphi/h}\|_{L^2(x_d>0)}.$$

Proposition 3.3. *Assume that a is smooth, (3), (4) hold and that the following hypotheses are fulfilled*

$$\frac{\partial \varphi}{\partial x_d} > 0 \text{ on } \{x_d = 0\} \cap B_\kappa,$$

$$1 > a^2 \text{ on } \{x_d = 0\} \cap B_\kappa,$$

$$(1 - a^2)\left(\frac{\partial \varphi}{\partial x_d}\right)^2 > a^2 \left[r\left(x, \frac{\partial \varphi}{\partial x'}\right) - a^2 \left(\frac{\partial \varphi}{\partial x_0}\right)^2 \right] \text{ on } \{x_d = 0\} \cap B_\kappa.$$

Then there exists $C, h_0 > 0$ such that for any $h \in (0, h_0)$, $g_1 \in \mathscr{C}^\infty(\{x_d = 0\})$, and any $g \in \mathscr{C}_0^\infty(B_\kappa)$ such that

$$\partial_{x_d} g + ia\partial_{x_0} g - bg = g_1 \text{ on } \{x_d = 0\},$$

the following inequality holds:

$$\|ge^{\varphi/h}\|_{L^2(x_d>0)} + \|h(\partial_x g)e^{\varphi/h}\|_{L^2(x_d>0)} + |h(\partial_{x'} g)e^{\varphi/h}| \leq C\left(h^{-1/2}\|h^2 P(g)e^{\varphi/h}\|_{L^2(x_d>0)} + |hg_1 e^{\varphi/h}|\right).$$

However, due to the low regularity of the function a here, we cannot apply this result directly and shall derive the following suitable result by a perturbation argument.

Corollary 3.4. *Assume that (3) and (4) hold. If a is a function such that $a(x) \xrightarrow[x \to 0]{} 0$,*

$$\frac{\partial \varphi}{\partial x_d} > 0 \text{ on } \{x_d = 0\} \cap B_\kappa$$

then, there exists $\kappa, C, h_0 > 0$, such that, for any $h \in (0, h_0)$, $g_1 \in L^2(\{x_d = 0\})$ and any $g \in H^1(\mathbb{R}^n)$ supported in B_κ which satisfies

$$P(g) \in L^2(\mathbb{R}^n) \text{ and } \partial_{x_d} g + ia\partial_{x_0} g - bg = g_1 \text{ on } \{x_d = 0\},$$

the following inequality holds:

$$\|ge^{\varphi/h}\|_{L^2(x_n>0)} + \|h(\partial_x g)e^{\varphi/h}\|_{L^2(x_n>0)} \leq C\left(h^{-1/2}\|h^2 P(g)e^{\varphi/h}\|_{L^2(x_d>0)} + |hg_1 e^{\varphi/h}|\right).$$

Proof. We apply Proposition 3.3 with $\overline{a} := 0$ and $\overline{g_1} := g_1 - ia\partial_{x_0} g$ to get, after a standard approximation argument,

$$\begin{aligned}
\|ge^{\varphi/h}\|_{L^2(x_n>0)} + \|h(\partial_x g)e^{\varphi/h}\|_{L^2(x_n>0)} + |h(\partial_{x'} g)e^{\varphi/h}| &\leq Ch^{-1/2}\|h^2 P(g)e^{\varphi/h}\|_{L^2(x_d>0)} \\
&+ C\left(|hg_1 e^{\varphi/h}| + |ha\partial_{x_0} ge^{\varphi/h}|\right) \\
&\leq Ch^{-1/2}\|h^2 P(g)e^{\varphi/h}\|_{L^2(x_d>0)} \\
&+ C|hg_1 e^{\varphi/h}| + h\, C\sup_{B_\kappa}|a|\, |\partial_{x_0} ge^{\varphi/h}|.
\end{aligned}$$

Choosing now κ sufficiently small so that $C\sup_{B_\kappa}|a| \leq 1/2$, we get the desired estimate. $\square$

We finally deduce from Theorem 2 the analog of the Carleman estimates of Propositions 3.1 , 3.2 and Cororally 3.4.

Corollary 3.5. *Assume that* (3) *and* (4) *hold. If* $\rho > 1/2$, $a \in \mathscr{C}^\rho(\{x_n = 0,\ x_1 < 0\})$ *such that* $a(x) \xrightarrow[x_1 \to 0]{} 0$,

$$\left(\frac{\partial \varphi}{\partial x_d} > 0 \text{ on } \{x_d = 0\} \cap B_\kappa\right) \text{ and } |\partial_{x'}\varphi(0)| \le \varepsilon \partial_{x_d}\varphi(0)$$

then, for any ε *sufficiently small, there exists* $\kappa, C, h_0 > 0$, *such that, for any* $h \in (0, h_0)$, $g_1 \in L^2(x_1 < 0)$ *and any* $g \in H^1(\mathbb{R}^n)$ *supported in* B_κ *which satisfies*

$$P(g) \in L^2(\mathbb{R}^n) \text{ and } \begin{cases} g = 0 & \text{on } \{x_d = 0, x_1 > 0\}, \\ \partial_{x_d} g + ia\partial_{x_0} g - bg = g_1 & \text{on } \{x_d = 0, x_1 < 0\}, \end{cases}$$

the following inequality holds:

$$\|ge^{\varphi/h}\|_{L^2(x_n>0)} + \|h(\partial_x g)e^{\varphi/h}\|_{L^2(x_n>0)} \le C\left(h^{-1/2}\|h^2 P(g)e^{\varphi/h}\|_{L^2(x_d>0)} + |hg_1 e^{\varphi/h}|_{L^2(x_1<0)}\right).$$

Proof. One applies Theorem 2 (with $n = d+1$) with $\overline{g_0} := 0$, $\overline{g_1} := g_1 - ia\partial_{x_0} g + bg$ and g such that

$$\begin{cases} g = \overline{g_0} & \text{on } \{x_d = 0, x_1 > 0\}, \\ \partial_{x_d} g = \overline{g_1} & \text{on } \{x_d = 0, x_1 < 0\}. \end{cases}$$

We get that $\|ge^{\varphi/h}\|_{H^1(x_d>0)} + |ge^{\varphi/h}|_{1/2} + |h(\partial_{x_d} g)e^{\varphi/h}|_{-1/2}$ is bounded by

$$\begin{aligned} &C\left(h^{-1/2}\|h^2 P(g)e^{\varphi/h}\|_{L^2(x_d>0)} + |hg_1 e^{\varphi/h}|_{H_{sc}^{-1/2}(x_1<0)}\right) \\ &+C\left(|ha(\partial_{x_0} g)e^{\varphi/h}|_{H_{sc}^{-1/2}(x_1<0)} + |hbge^{\varphi/h}|_{H_{sc}^{-1/2}(x_1<0)}\right). \end{aligned}$$

Moreover, we have

$$|hbge^{\varphi/h}|_{H^{-1/2}(x_1<0)} \le |hbge^{\varphi/h}|_{L^2(x_1<0)} \le h \sup_{B_\kappa} |b|\ |ge^{\varphi/h}|_{H^1(x_1<0)}. \tag{74}$$

On the other hand, one has $(\partial_{x_0} g)e^{\varphi/h} \in H^{-1/2}(x_1 < 0)$ so there exists $u \in H^{-1/2}(\mathbb{R}^{n-1})$ such that $u_{|x_1<0} = (\partial_{x_0} g)e^{\varphi/h}$.
Denoting $\overline{a} := 1_{\{x_1<0\}} a \in \mathscr{C}^\rho(\mathbb{R}^{n-1})$, one now has, provided $\rho > 1/2$,

$$\overline{a}u \in H^{-1/2}(\mathbb{R}^{n-1}), \tag{75}$$

using standard continuity result (see e.g. [27, Corollary p. 143]).
Since $\overline{a}u = 1_{\{x_1<0\}} a(\partial_{x_0} g)e^{\varphi/h} = \overline{a}(\partial_{x_0} g)e^{\varphi/h}$ and according to the definition of the $H_{sc}^{-1/2}(x_1 < 0)$ norm, one consequently has

$$|a(\partial_{x_0} g)e^{\varphi/h}|_{H_{sc}^{-1/2}(x_1<0)} \le |\overline{a}(\partial_{x_0} g)e^{\varphi/h}|_{-1/2}.$$

Using [27, Corollary p. 143] again, we deduce, for $1/2 < \rho' < \rho$,

$$|a(\partial_{x_0} g)e^{\varphi/h}|_{H_{sc}^{-1/2}(x_1<0)} \le |\overline{a}(\partial_{x_0} g)e^{\varphi/h}|_{-1/2} \le C\|\overline{a}\chi(./\kappa)\|_{\mathscr{C}^{\rho'}}\ |(\partial_{x_0} g)e^{\varphi/h}|_{H_{sc}^{-1/2}(x_1<0)}, \tag{76}$$

where χ is some regular cut-off function supported in B_2 such that $\chi = 1$ on B_1 and, for any $x \in \mathbb{R}^{n-1}$, $\chi(./\kappa)(x) = \chi(x/\kappa)$.
One may now apply Lemma C.1 and get that

$$|a(\partial_{x_0} g)e^{\varphi/h}|_{H_{sc}^{-1/2}(x_1<0)} \le C\kappa^{\rho-\rho'}\|a\|_{\mathscr{C}^\rho}|(\partial_{x_0} g)e^{\varphi/h}|_{H_{sc}^{-1/2}(x_1<0)}. \tag{77}$$

Summing up (74) and (77), we deduce that, for h and κ sufficiently small,

$$\begin{aligned} &\|ge^{\varphi/h}\|_{H^1(x_d>0)} + |ge^{\varphi/h}|_{1/2} + |h(\partial_{x'} g)e^{\varphi/h}|_{-1/2} \\ &\le C\left(h^{-1/2}\|h^2 P(g)e^{\varphi/h}\|_{L^2(x_d>0)} + |hg_1 e^{\varphi/h}|_{H_{sc}^{-1/2}(x_1<0)}\right). \end{aligned}$$

Finally, writing $g = (ge^{\varphi/h})e^{-\varphi/h}$, we have

$$\|ge^{\varphi/h}\|_{L^2(x_d>0)} + \|h(\partial_x g)e^{\varphi/h}\|_{L^2(x_d>0)} \le C\|ge^{\varphi/h}\|_{H^1(x_d>0)}$$

and the sought result is proved. ∎

3.2 Proof of Proposition 1.2

We follow the method of Lebeau and Robbiano (see [22, Paragraph 3.B]). We here simply insist on the points that differ in our context.

Interpolation inequality away from the boundary Defining $X_d = (-3/4, 3/4) \times \Omega_d$ with $\Omega_d = \{x \in \Omega; d(x, \partial\Omega) \geqslant d\}$, we first recall an interpolation inequality for system (2) away from the boundary.

Lemma 3.6. *There exists $C > 0$ and $\tau_0 \in (0,1)$ such that for any $\tau \in [0, \tau_0]$ and for any function v solution of* (2)*, the following inequality holds*

$$\|v\|_{H^1(X_d)} \leqslant C \left(\|\Delta_X v\|_{L^2(X)} + \|v\|_{L^2(\partial X_N^\delta)} + \|\partial_\nu v\|_{L^2(\partial X_N^\delta)} \right)^\tau \|v\|_{H^1(X)}^{1-\tau}.$$

Proof. We first have, for any function v solution of (2),

$$\|v\|_{H^1(X_d)} \leqslant C \left(\|\Delta_X v\|_{L^2(X)} + \|\theta v\|_{L^2(\partial X)} + \|\theta \partial_\nu v\|_{L^2(\partial X)} \right)^\tau \|v\|_{H^1(X)}^{1-\tau}$$

for $\theta \in \mathscr{C}_0^\infty(\partial X)$ any non-trivial function.
Indeed, if $v = 0$ on ∂X, this result is contained in [22, Paragraph 3.B, equation (45)]. The proof of this slightly more general interpolation result is absolutely identical and we shall not detail it here (see also [22, Paragraph 3.B, Lemme 3] for a local version of this estimate).
It suffices now to choose θ supported in ∂X_N^δ to obtain the Lemma. ☐

Interpolation method near the boundary We now use the Carleman estimates described in Section 3.1 to prove estimates near the boundary.

First of all, we begin by the definition of the phase function inspired by [22]. We put $\varphi = f(\psi)$ where $f(t) = e^{\beta t}$ and, writing $x = (x_0, y)$,

$$\forall x \in X, \ \psi(x) = \psi_0(x_0) + \psi_1(y)$$

where, for some $d > 0$, ψ_1 is such that $\partial_\nu \psi_1(y) > 0$ if $d(y, \partial\Omega) < 3d$ (for ∂_ν a vector field defined in some neighbourhood of $\partial\Omega$ which extends the normal derivative on $\partial\Omega$) and

$$\psi_1(y) = \begin{cases} d(y, \partial\Omega) & \text{if } d(y, \partial\Omega) \leqslant 2d, \\ 3d & \text{if } d(y, \partial\Omega) \geqslant 3d, \end{cases}$$

and, for some $\varepsilon > 0$, ψ_0 is an even function such that $\psi_0'(x_0) < 0$ if $x_0 > 1/2$ with

$$\psi_0(x_0) = \begin{cases} 0 & \text{if } x_0 \in [0, 1/2], \\ \varepsilon(1 - 2x_0) & \text{if } x_0 \in [3/4, 1). \end{cases}$$

We will now show that we can apply our Carleman estimates to our function φ.
First, it is classical that the function $\varphi = f(\psi)$ satisfies Hörmander hypoellipticity condition (4) for some $\beta > 0$ large enough. We refer the reader to Lemme 3 in [22, Paragraph 3.B] for a proof.
On the other hand, since $\partial_\nu \varphi = f'(\psi)\partial_\nu \psi$ and $\partial_{x'} \varphi = f'(\psi)\partial_{x'}\psi$, it is clear that

$$|\partial_{x'}\varphi| \leq C\varepsilon |\partial_\nu \varphi| \text{ on } \partial X$$

and the hypotheses of Theorem 2 are fullfilled for ε sufficiently small.

A finite partition of unity on $\partial\Omega$ combined with Propositions 3.1, 3.2 and Corollaries 3.4, 3.5 may now show that

$$\begin{aligned} &\|e^{\varphi/h} w\|_{L^2(X)}^2 + \|e^{\varphi/h} h \partial_x w\|_{L^2(X)}^2 \\ &\leq C \Big(h^3 \|e^{\varphi/h} P w\|_{L^2(X)}^2 + h^2 \|e^{\varphi/h}(\partial_\nu - ia\partial_{x_0} + b) w\|_{L^2(\partial X_N)}^2 \\ &\quad + \|e^{\varphi/h} w\|_{L^2(\partial X_N^\delta)} + \|h e^{\varphi/h} \partial_\nu w\|_{L^2(\partial X_N^\delta)} \Big) \end{aligned} \tag{78}$$

for any $w \in H^1(X)$ supported in some small neighbourhood W of ∂X which also satisfies $Pw \in L^2(X)$, $w = 0$ on ∂X_D and $\partial_\nu w \in L^2(\partial X_N)$.
Indeed, one first chooses the partition of unity (θ_i) on some neighborhood of $\partial\Omega$ such that any element of this partition θ lies in one of the following cases:

1. $\mathrm{supp}(\theta) \cap \partial\Omega \subset \partial\Omega_D$,
2. $\mathrm{supp}(\theta) \cap \partial\Omega \subset \{y \in \partial\Omega_N; a(y) < 2\delta\}$,
3. $\mathrm{supp}(\theta) \cap \partial\Omega \subset \{y \in \partial\Omega_N; a(y) > \delta\}$,
4. $\mathrm{supp}(\theta) \cap \partial\Omega \subset \partial\Omega_D \cup \{y \in \partial\Omega_N; a(y) < 2\delta\}$ and θ supported in a neighborhood of Γ.

Next, for δ and $\mathrm{supp}(\theta)$ chosen sufficiently small, one defines $g = \theta\tilde{v}$. Working in local coordinates such that (69) and (72) hold, we may apply to function g

- Proposition 3.2 in case 1,
- Corollary 3.4 in case 2,
- Proposition 3.1 in case 3,
- Corollary 3.5 in case 4,

and, summing up these inequalities, we directly get the estimate (78). Note in particular that the estimates of Propositions 3.2, 3.3 and 3.1 can be applied to w, using a standard approximation argument.

To get now our interpolation inequality, we define, for $r_1 < r_1' < r_2 < r_2' < r_3 < r_3'$, the sets (see Figure 2)

$$V = \{x \in X; r_1 \le \psi(x) \le r_3'\}$$

and, for $j = 1, 2, 3$,

$$V_j = \{x \in X; r_j \le \psi(x) \le r_j'\}.$$

As in [22], we choose $r_1 = -2d$, $r_1' = -d$, $r_2 = 0$, $r_2' = d$, $r_3 = 3/2d$, $r_3' = 2d$ so that, using the definition of ψ,

$$(-1/2, 1/2) \times (\Omega \backslash \Omega_d) \subset V_2, \qquad V_3 \subset (-1,1) \times \Omega_d. \tag{79}$$

Moreover, using that $\inf \psi_0 = -\varepsilon$ and $\sup \psi_1 = 3d$, one has $V \subsetneq X$ for d sufficiently small so that $-5d > -\varepsilon$.

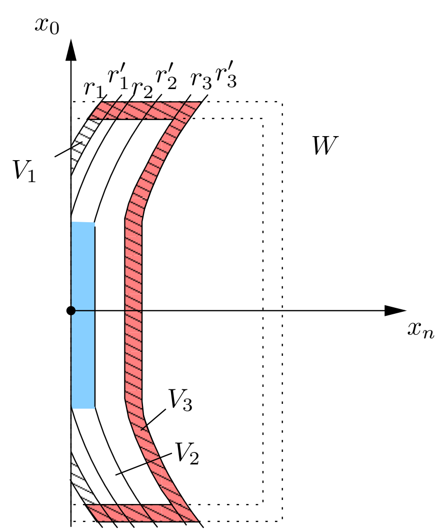

Figure 2: Interpolation sets V_1, V_2, V_3 and level sets associated to $(r_i)_{1\le i\le 3}$, $(r_i')_{1\le i\le 3}$.

We now apply the interpolation method (see Lemme 3 in [22, Paragraph 3.B]). We choose χ a $\mathscr{C}^\infty$ function supported in V and W such that $\chi = 1$ for $\psi(x) \in [r_1', r_3]$ and we apply the Carleman estimate (78) to $w = \chi\tilde{v}$ where $\tilde{v}$ is the solution of (73).
We write $Pw = \chi P\tilde{v} + [P, \chi]\tilde{v}$ and we note that $[P, \chi]$ is a differential operator of order 1 supported in $V_1 \cup V_3$. Hence,

$$\|e^{\varphi/h} Pw\|_{L^2(X)} \leq \|e^{\varphi/h} v_0\|_{L^2(V)} + C\|e^{\varphi/h}\tilde{v}\|_{H^1(V_1)} + C\|e^{\varphi/h}\tilde{v}\|_{H^1(V_3)}.$$

Analogously, using trace estimates on V_1 and V_3, we have

$$\|e^{\varphi/h}(\partial_\nu - ia\partial_{x_0} + b)w\|_{L^2(\partial X_N)} \leq \|e^{\varphi/h} v_1\|_{L^2(\partial X_N \cap V)} + C\|e^{\varphi/h}\tilde{v}\|_{H^1(V_1)} + C\|e^{\varphi/h}\tilde{v}\|_{H^1(V_3)}.$$

Summing up, we obtain by (78)

$$\begin{aligned} \|e^{\varphi/h}\tilde{v}\|_{L^2(V_2)} + \|e^{\varphi/h}\partial_x\tilde{v}\|_{L^2(V_2)} \quad &\leqslant \quad C\Big(\|e^{\varphi/h}v_0\|_{L^2(V)} + \|e^{\varphi/h}v_1\|_{L^2(\partial X_N \cap V)} \\ &+ \quad \|e^{\varphi/h}\tilde{v}\|_{L^2(\partial X_N^\delta \cap V)} + \|e^{\varphi/h}\partial_\nu\tilde{v}\|_{L^2(\partial X_N^\delta \cap V)} \\ &+ \quad \|e^{\varphi/h}\tilde{v}\|_{L^2(V_1)} + \|e^{\varphi/h}\partial_x\tilde{v}\|_{L^2(V_1)} \\ &+ \quad \|e^{\varphi/h}\tilde{v}\|_{L^2(V_3)} + \|e^{\varphi/h}\partial_x\tilde{v}\|_{L^2(V_3)}\Big) \end{aligned}$$

so that, using the definition of φ,

$$e^{f(r_2)/h}\|\tilde{v}\|_{H^1(V_2)} \leqslant Ce^{f(r_1')/h}\|\tilde{v}\|_{H^1(V_1)}$$
$$+Ce^{f(r_3')/h}\Big(\|v_0\|_{L^2(X)} + \|v_1\|_{L^2(\partial X_N)} + \|\tilde{v}\|_{L^2(\partial X_N^\delta)} + \|\partial_\nu\tilde{v}\|_{L^2(\partial X_N^\delta)} + \|\tilde{v}\|_{H^1(V_3)}\Big)$$

and, using (79) and coming back to the solution v of (2),

$$e^{f(r_2)/h}\|v\|_{H^1(Y/X_d)} \leqslant Ce^{f(r_1')/h}\|v\|_{H^1(X)}$$
$$+Ce^{f(r_3')/h}\Big(\|v_0\|_{L^2(X)} + \|v_1\|_{L^2(\partial X_N)} + \|v\|_{L^2(\partial X_N^\delta)} + \|\partial_\nu v\|_{L^2(\partial X_N^\delta)} + \|v\|_{H^1(X_d)}\Big).$$

In the same way as it was done in [22, Paragraph 3.B] and since $f(r_1') < f(r_2) < f(r_3')$, one may deduce after an optimization in $h \in (0, h_0)$ that there exists $\tau_0 \in (0, 1)$ such that

$$\|v\|_{H^1(Y\setminus X_d)}$$
$$\leqslant C\Big(\|v_0\|_{L^2(X)} + \|v_1\|_{L^2(\partial X_N)} + \|v\|_{L^2(\partial X_N^\delta)} + \|\partial_\nu v\|_{L^2(\partial X_N^\delta)} + \|v\|_{H^1(X_d)}\Big)^{\tau_0}\|v\|_{H^1(X)}^{1-\tau_0}.$$

If one combines this result with Lemma 3.6 and taking account that $\partial_\nu v = -ia(x)\partial_{x_0}v + v_1$ on ∂X_N^δ, we get the sought result of Proposition 1.2.

3.3 End of the proof

3.3.1 Preliminary settings

First of all, let us recall that the system (1) possesses a unique solution. We define the Hilbert space $H = H_D^1(\Omega) \times L^2(\Omega)$ (where $H_D^1(\Omega) = \{u \in H^1(\Omega); u = 0 \text{ on } \partial\Omega_D\}$) equipped with the norm

$$\|(u_0, u_1)\|_H^2 = \int_\Omega |\partial_x u_0|^2 + |u_1|^2 dx.$$

We also define the unbounded operator $\mathcal{A}$ on H by

$$\mathcal{A} = \begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix}$$

with domain
$$\mathcal{D}(\mathcal{A})=\{u=(u_0,u_1)\in H;\mathcal{A}u\in H \text{ and } \partial_\nu u_0+au_1=0\}.$$
It is clear that system (1) can be rewritten in terms of the abstract problem
$$\left\{\begin{array}{l}\partial_t U(t)=\mathcal{A}U(t),\\ U(0)=\begin{pmatrix}u_0\\ u_1\end{pmatrix},\end{array}\right.$$
where $U=\begin{pmatrix}u\\ \partial_t u\end{pmatrix}$.

Moreover, $\mathcal{A}$ is a monotone operator. Indeed, an integration by parts gives, for any $u=\begin{pmatrix}u_0\\ u_1\end{pmatrix}\in\mathcal{D}(\mathcal{A})$,
$$\operatorname{Re}(\mathcal{A}u,u)_H=\int_{\partial\Omega_N}\overline{\partial_\nu u_0}u_1 d\sigma=-\int_{\partial\Omega_N}a|u_1|^2d\sigma\leqslant 0.$$
On the other hand, $\mathcal{A}-I$ is an isomorphism from $\mathcal{D}(\mathcal{A})$ to H. Indeed, one easily obtains for $\begin{pmatrix}f_0\\ f_1\end{pmatrix}\in H$,
$$(\mathcal{A}-I)\begin{pmatrix}u_0\\ u_1\end{pmatrix}=\begin{pmatrix}f_0\\ f_1\end{pmatrix}\Leftrightarrow\left\{\begin{array}{l}-u_0+\Delta u_0=f_0+f_1,\\ u_1=u_0+f,\end{array}\right.$$
and the bijectivity is granted by Lax-Milgram noting that the bilinear form
$$(u,v)\in H^1_D(\Omega)^2\mapsto\int_\Omega(\partial_x u.\partial_x\bar{v}+u\bar{v})dx+\int_{\partial\Omega_N}au\bar{v}d\sigma$$
is coercive. Hence, Hille-Yoshida theorem gives that $\mathcal{A}$ generates a strongly continuous semigroup on H and hence the existence and unicity of solutions to problem (1).

3.3.2 Proofs of Proposition 1.1 and Theorem 1

We consequently focus on the equation
$$(\mathcal{A}-i\lambda I)\begin{pmatrix}u_0\\ u_1\end{pmatrix}=\begin{pmatrix}f_0\\ f_1\end{pmatrix}.$$
We will write $R(\mu)=(\mathcal{A}-\mu I)^{-1}$ when it is defined.
One has the following system
$$\left\{\begin{array}{ll}(\Delta+\lambda^2)u_0=(i\lambda f_0+f_1) & \text{in } \Omega,\\ (\partial_\nu+i\lambda a(x))u_0=-a(x)f_0 & \text{on } \partial\Omega_N,\\ u_0=0 & \text{on } \partial\Omega_D,\end{array}\right. \tag{80}$$
and we introduce $v(x_0,x)=e^{\lambda x_0}u_0(x)$,
$$X=(-1,1)\times\Omega,\partial X_N=(-1,1)\times\partial\Omega_N \text{ and } \partial X_D=(-1,1)\times\partial\Omega_D$$
so that v is solution of
$$\left\{\begin{array}{ll}\Delta_X v=e^{\lambda x_0}(i\lambda f_0+f_1) & \text{in } X,\\ (\partial_\nu+ia(x)\partial_{x_0})v=e^{\lambda x_0}(-a(x)f_0) & \text{on } \partial X_N,\\ v=0 & \text{on } \partial X_D.\end{array}\right.$$
Applying Proposition 1.2 to $v_0:=e^{\lambda x_0}(i\lambda f_0+f_1)$ and $v_1:=e^{\lambda x_0}(-a(x)f_0)$, one may get the estimate, for any $\lambda\in\mathbb{R}$,
$$\|u_0\|_{H^1(\Omega)}\leqslant Ce^{C|\lambda|}\left(\|f_0\|_{H^1(\Omega)}+\|f_1\|_{L^2(\Omega)}+\|u_0\|_{L^2(\partial\Omega_N^\delta)}\right), \tag{81}$$

where we have noted $\partial\Omega_N^\delta = \{x \in \partial\Omega_N ; a(x) > \delta\}$.
If $\|u_0\|_{L^2(\partial\Omega_N^\delta)} \leqslant \|f_0\|_{H^1(\Omega)} + \|f_1\|_{L^2(\Omega)}$, one consequently has

$$\|u_0\|_{H^1(\Omega)} \leqslant Ce^{C|\lambda|}\left(\|f_0\|_{H^1(\Omega)} + \|f_1\|_{L^2(\Omega)}\right). \tag{82}$$

Else, we have the following estimate

$$\|u_0\|_{H^1(\Omega)} \leqslant Ce^{C|\lambda|}\|u_0\|_{L^2(\partial\Omega_N^\delta)}. \tag{83}$$

Using that

$$\int_\Omega \overline{u_0}(-\Delta - \lambda^2)u_0 dx = -\lambda^2\|u_0\|_{L^2(\Omega)} + \int_\Omega |\partial_x u_0|^2 dx - \int_{\partial\Omega_N} \overline{u_0}\partial_\nu u_0 d\sigma$$

and, since $(\Delta + \lambda^2)u_0 = i\lambda f_0 + f_1$ and $\partial_\nu u_0 + ia\lambda u_0 = f_0$, we get, taking the imaginary part of this identity,

$$|\lambda| \int_{\partial\Omega_N} a|u_0|^2 d\sigma \leqslant (|\lambda|\|f_0\|_{H^1(\Omega)} + \|f_1\|_{L^2(\Omega)})\|u_0\|_{L^2(\Omega)}.$$

Consequently (83) and Young inequality give us

$$\|u_0\|_{H^1(\Omega)} \leqslant Ce^{C|\lambda|}(\|f_0\|_{H^1(\Omega)} + \|f_1\|_{L^2(\Omega)})$$

and one also gets the estimate (82).
Since $u_1 = f_0 + i\lambda u_0$, we get in both cases the aditional estimate

$$\|u_1\|_{L^2(\Omega)} \leqslant \|f_0\|_{H^1(\Omega)} + |\lambda|\|u_0\|_{H^1(\Omega)} \leqslant Ce^{C|\lambda|}\left(\|f_0\|_{H^1(\Omega)} + \|f_1\|_{L^2(\Omega)}\right).$$

We consequently have proved that, for any λ sufficiently large,

$$\|u_0\|_{H^1(\Omega)} + \|u_1\|_{L^2(\Omega)} \leqslant Ce^{C|\lambda|}\left(\|f_0\|_{H^1(\Omega)} + \|f_1\|_{L^2(\Omega)}\right)$$

and that $\mathcal{A} - i\lambda I$ is one-to-one.
Since $\mathcal{D}(\mathcal{A}) \hookrightarrow H$ is compact, the Fredholm alternative (see e.g. [6, Théorème VI.6]) gives us that $\mathcal{A}$ is onto (because $(\mathcal{A} - i\lambda I)u = f \Leftrightarrow (I + (i\lambda - 1)R(1))u = R(1)f$ where $R(1)$ is a compact operator) and finally that $\mathcal{A} - i\lambda I$ is an isomorphism. Moreover,

$$|\lambda| \geqslant \lambda_1 \Rightarrow \|R(i\lambda)\|_{H\to H} \leqslant Ce^{C|\lambda|}.$$

On the other hand, the spectrum of $\mathcal{A}$ is discrete. Furthermore, for any $\lambda \in \mathbb{R}$, $\mathcal{A} - i\lambda I$ is one-to-one. Indeed, if

$$(\mathcal{A} - i\lambda I)\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = 0$$

then, using a special case of (80), one has

$$\int_\Omega (\Delta + \lambda^2)u_0\overline{u_0}dx = 0$$

so that, using an integration by parts and the boundary conditions of u_0,

$$\int_\Omega (-|\partial_x u_0|^2 + \lambda^2|u_0|^2)dx - i\lambda \int_{\partial\Omega_N} a|u_0|^2 d\sigma = 0.$$

Taking the imaginary part of this expression, we get $\lambda \times au_0 = 0$ in $\partial\Omega_N$ and then $u_0 = 0$ on $\partial\Omega_N^\delta$. Since u_0 also satisfies system (80) with $f_0 = 0$ and $f_1 = 0$, the unique continuation estimate (81) gives us $u_0 = 0$ and hence $(u_0, u_1) = 0$.

Using again that $\mathcal{D}(\mathcal{A}) \hookrightarrow H$ is compact and the Fredholm alternative, we get that $i\mathbb{R} \subset \rho(\mathcal{A})$. Since $\lambda \in \rho(\mathcal{A}) \mapsto R(\lambda)$ is continuous, there consequenty exists some constant $C > 0$ so that

$$|\lambda| \leqslant \lambda_1 \Rightarrow \|R(i\lambda)\|_{H\to H} \leqslant Ce^{C|\lambda|},$$

which concludes our proof of Proposition 1.1.
Theorem 1 is then an immediate application of [1, Theorem A] (see also Théorème 3 in [7]).

4 Comments and further applications

Regularity of the function a. We begin by some considerations on the sufficient regularity of the function a. Using paradifferential calculus, is it not hard to get that a can be chosen in the Besov space $B^{1/2}_{\infty,2}(\partial\Omega_N) \hookrightarrow L^\infty(\partial\Omega_N)$. Indeed, the main point is to get continuity estimate analogous to Equations (75), (76) in the proof of Corollary 3.5 and this can be done in a standard way by the use of paraproduct decomposition (introduced in [4]).
Moreover, some regularity result of solutions to the Zaremba problem would allow us to extend our Theorem 1 to less regular functions a. More precisely, given $f \in L^2(\Omega)$, $f_0 \in L^2(\partial\Omega_D)$, $f_1 \in L^2(\partial\Omega_N)$, if any $u \in H^1(\Omega)$ such that

$$\begin{cases} \Delta u = f & \text{in } \Omega, \\ u = f_0 & \text{on } \partial\Omega_D, \\ \partial_\nu u = f_1 & \text{on } \partial\Omega_N, \end{cases}$$

belongs in fact to $H^{3/2-\epsilon}(\Omega)$ for any $\epsilon > 0$, we would be able to get our result for $a \in \mathscr{C}^\rho(\partial\Omega_N)$ with $\rho > 0$. Unfortunately, this regularity result seems to be known only in dimension 2 (see, e.g., the book [11]).

Other geometric cases. On the other hand, the Carleman estimate given in Theorem 2 is local in a neighborhood of $(-1,1)\times\Gamma$, thus we can use it in other geometric cases, patching this estimate with other Carleman estimates either to prove a global Carleman estimate or to prove an interpolation inequality in the spirit of Proposition 1.2.

In particular, Theorem 1 remains valid if Ω is replaced by (M, g) a smooth compact riemannian manifold with boundary and Γ is a smooth submanifold of ∂M of codimension 1. Note that Γ is not necessarily connected (see Figure 3).
Indeed, it is sufficient to work on each connected component of M and to notice that our proof remains valid on each component.

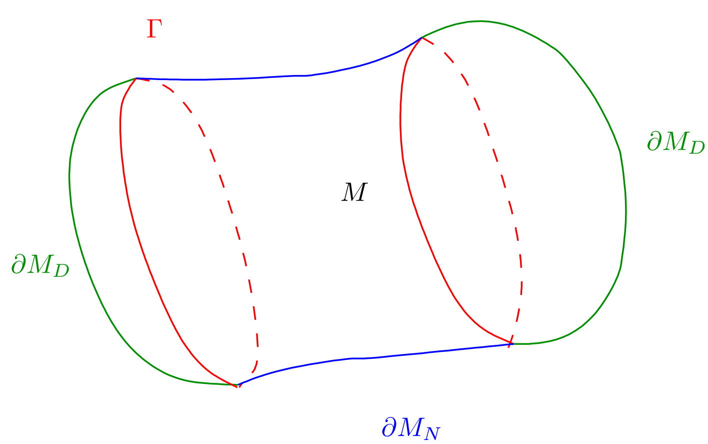

Figure 3: A configuration example where Γ is not connected.

Control of the heat equation. With the Carleman estimate found (Theorem 2), we can also prove some null controllability results for the heat equation with the Zaremba boundary condition. We give the result without detailled proof.

Let ω an open subset of Ω such that $\overline{\omega} \neq \Omega$ and $T > 0$. Then, for all $u_0 \in L^2(\Omega)$, there exists $f \in L^2((0,T)\times\omega)$ such that the solution u of the following system

$$\begin{cases} \partial_t u - \Delta u = 1_\omega f & \text{in } \Omega \times \mathbb{R}^+, \\ u = 0 & \text{on } \partial\Omega_D \times \mathbb{R}^+, \\ \partial_\nu u = 0 & \text{on } \partial\Omega_N \times \mathbb{R}^+, \\ u_{|t=0} = u_0 & \text{in } \Omega, \end{cases}$$

satisfies $u(T,.) = 0$.

We refer the reader to the survey by Le Rousseau and Lebeau [17, Sections 5-6] where a strategy of proof is explained in details.

First of all, it is sufficient to have the following interpolation inequality (see [17, Theorem 5.3]). Let $X = (-1,1) \times \Omega$ and $Y = (-1/2, 1/2) \times \Omega$. Then, there exists $C > 0$ and $\tau \in (0,1)$ such that, for any $v \in H^2(X)$ such that $v = 0$ on $(-1,1) \times \partial\Omega_D$, $\partial_\nu v = 0$ on $(-1,1) \times \partial\Omega_N$ and $v(-1,.) = 0$ on Ω,

$$\|v\|_{H^1(Y)} \leq C \|v\|_{H^1(X)}^{1-\tau} \left(\|(\partial_{x_0}^2 + \Delta) v\|_{L^2(X)} + \|\partial_{x_0} v(0,.)\|_{L^2(\omega)} \right)^\tau .$$

This interpolation estimate is analogous but different from the one of Proposition 1.2. Nevertheless, we can prove it in the same way: the main novelty is to obtain a local interpolation inequality in a neighborhood of $(-1/2,1/2) \times \Gamma$ and this can be done exactly as in Section 3.2.

With this interpolation inequality and following [17, Theorem 5.4], we can then obtain an inequality on sums of eigenfunctions.

Let $(\phi_j)_{j\in\mathbb{N}^*}$ be a orthonormal basis of eigenfunctions of the Laplacian with the Zaremba boundary condition and $0 < \mu_1 \leq \mu_2 \leq ...$ the associated eigenvalues. Then, there exists $C > 0$ such that, for all complex sequences $(\alpha_j)_{j\in\mathbb{N}^*}$ and all $\mu > 0$,

$$\sum_{\mu_j \leq \mu} |\alpha_j|^2 \leq C e^{C\sqrt{\mu}} \int_\omega \Big| \sum_{\mu_j \leq \mu} \alpha_j \phi_j \Big|^2 dx.$$

Following the strategy of [17, Section 6], one is now able to construct a control function f.

A Symbolic Calculus

We use the metric definition given by Hörmander and, if g is a metric and m a g-continuous function, the notation $S(m,g)$ of symbol spaces (see definitions 18.4.1, 18.4.7 and 18.4.2 in [14]).

We denote $\langle \eta \rangle_\varepsilon^2 = |\eta|^2 + \varepsilon^2$ and g the metric

$$g = dx'^2 + \frac{d\xi_1^2}{\langle \xi' \rangle_\varepsilon^2} + \frac{d\xi''^2}{\langle \xi'' \rangle_\varepsilon^2}.$$

Lemma A.1. *g is a metric slowly varying, semi-classical σ-temperate, uniformly with respect to ε if $h \leq \varepsilon$.*

The weights $\langle \xi' \rangle$, $\langle \xi' \rangle_\varepsilon$ and $\langle \xi'' \rangle_\varepsilon$ are g-continuous and semi-classical σ, g-temperate uniformly with respect to ε if $h \leq \varepsilon$.

Moreover, we have

$$\mathfrak{h}^2(x', \xi') := \sup_{(y',\eta')} \left(\frac{g_{(x',\xi')}(y',\eta')}{g^\sigma_{(x',\xi')}(y',\eta')} \right) = \langle \xi'' \rangle_\varepsilon^{-2}.$$

Remark 6. When a symbol $a(x,\xi)$ is quantified in the semi-classical sense, the usual symbol is $a_h(x,\xi) := a(x, h\xi)$.

If $a \in S(m,g)$ then $a_h \in S(m_h, g_h)$ where $m_h(x,\xi) := m(x,h\xi)$ and $(g_h)_{(x,\xi)}(dx, d\xi) := g_{(x,h\xi)}(dx, hd\xi)$.

We verify that

$$(g_h)^\sigma = h^{-2} (g^\sigma)_h$$

and the calculus is admissible if g_h is temperate.

In this case we say that g *is semi-classical temperate*. The condition on g read, there exists $C>0$ and $N>0$ such that for all x,y,z,ξ,η,ζ,

$$g_{(x,\xi)}(z,\zeta) \leq Cg_{(y,\eta)}(z,\zeta)(1+h^{-2}g^{\sigma}_{(x,\xi)}(x-y,\xi-\eta))^N$$

Analogously, we say that m *is semi-classical* σ, g*-temperate* if, there exists $C>0$ and $N>0$ such that, for all x,y,ξ,η,

$$m(x,\xi) \leq Cm(y,\eta)(1+h^{-2}g^{\sigma}_{(x,\xi)}(x-y,\xi-\eta))^N$$

On the other hand, if $\mathfrak{h}_h^2(x,\xi) := \sup\limits_{(y,\eta)}\left(\dfrac{(g_h)_{(x,\xi)}(y,\eta)}{(g_h)^{\sigma}_{(x,\xi)}(y,\eta)}\right)$, one has

$$\mathfrak{h}_h(x,\xi) = h\mathfrak{h}(x,h\xi).$$

Proof. We have $g^\sigma = \langle\xi'\rangle_\varepsilon^2 dx_1^2 + \langle\xi''\rangle_\varepsilon^2 dx''^2 + d\xi'^2$. If

$$\frac{|\xi_1-\eta_1|^2}{\langle\xi'\rangle_\varepsilon^2} + \frac{|\xi''-\eta''|^2}{\langle\xi''\rangle_\varepsilon^2} < \delta$$

where δ is small enough, then

$$\langle\xi'\rangle_\varepsilon^2 \sim \langle\eta'\rangle_\varepsilon^2,\ \langle\xi''\rangle_\varepsilon^2 \sim \langle\eta''\rangle_\varepsilon^2 \text{ and } \langle\xi'\rangle^2 \sim \langle\eta'\rangle^2$$

uniformly with respect to ε. We deduce that g is slowly varying and that $\langle\xi'\rangle_\varepsilon$, $\langle\xi''\rangle_\varepsilon$ and $\langle\xi'\rangle$ are g continuous.

To prove the temperance, the key point is the following estimate: there exists $C>0$ and $N>0$ such that

$$\forall\xi',\eta'\in\mathbb{R}^{n-1},\ \langle\eta''\rangle_\varepsilon^2 \leq C\langle\xi''\rangle_\varepsilon^2(1+h^{-2}|\xi'-\eta'|)^N \text{ and } \langle\eta'\rangle_\varepsilon^2 \leq C\langle\xi'\rangle_\varepsilon^2(1+h^{-2}|\xi'-\eta'|)^N. \tag{84}$$

Indeed, one has

$$|\eta|^2+\varepsilon^2 \leq 2(|\xi|^2+\varepsilon^2)+2|\xi-\eta|^2 \leq 2\langle\xi\rangle_\varepsilon^2(1+h^{-2}|\xi-\eta|^2)$$

if $h\leq\varepsilon$. The temperance is then a straightforward consequence of (84).

Moreover, it is also easy to see that

$$\mathfrak{h}^2(x',\xi') = \sup_{(y',\eta')}\left(\frac{|y'|^2+\frac{\eta_1^2}{\langle\xi'\rangle_\varepsilon^2}+\frac{|\eta''|^2}{\langle\xi''\rangle_\varepsilon^2}}{\langle\xi'\rangle_\varepsilon^2 y_1^2+\langle\xi''\rangle_\varepsilon^2|y''|^2+|\eta'|^2}\right) = \frac{1}{\langle\xi''\rangle_\varepsilon^2}.$$

since $\langle\xi''\rangle_\varepsilon \leq \langle\xi'\rangle_\varepsilon$ and equality is obtained for $y_1=1$, $y''=0$ and $\eta'=0$.

□

In the sequel, we will need the following version of the sharp semi-classical Gårding inequality.

Proposition A.2. *Let $a\in S(1,g)$ such that $a\geq 0$. Then there exists $C_\varepsilon>0$ and $h_\varepsilon>0$ such that*

$$\forall f\in L^2(\mathbb{R}^{n-1}),\forall h\in(0,h_\varepsilon);\mathrm{Re}(\mathrm{op}(a)f|f)_0 \geqslant -C_\varepsilon h|f|^2,$$

where $(.,.)_0$ is the standard scalar product on $L^2(\mathbb{R}^{n-1})$.

Proof. We first note that

$$\mathfrak{h}_h(x',\xi') = \frac{h}{\langle h\xi''\rangle_\varepsilon} \leq \frac{h}{\varepsilon}$$

and, since $a\in S(1,g)$,

$$\frac{\varepsilon}{h}a_h \in S\left(\frac{1}{\mathfrak{h}_h},g_h\right).$$

If $h \leq \varepsilon$ then $\mathfrak{h}_h \leq 1$ and the standard sharp Gårding inequality (see [14, Theorem 18.6.7]) applied to $\frac{\varepsilon}{h} a_h$ directly gives

$$\forall f \in L^2(\mathbb{R}^{n-1}), \forall h \in (0, \varepsilon); (\mathrm{op}_W(a) f|f)_0 \geqslant -C_\varepsilon h |f|^2,$$

where op_W denotes the Weyl quantified operator associated to a.
Moreover, one may classicaly write

$$\mathrm{op}_W(a) = \mathrm{op}(\tilde{a})$$

for some $\tilde{a}$ which satisfies

$$\tilde{a} = a + hb$$

where $b \in S(1, g)$. The result is then a straightforward consequence of the L^2 continuity of $\mathrm{op}(b)$. □

Lemma A.3. *Let $(a_\varepsilon)_{\varepsilon \in (0,1)}$ a family of $S(1, g)$ and*

$$M_\varepsilon = \sup_{(x', \xi') \in \mathbb{R}^{n-1} \times \mathbb{R}^{n-1}} |a_\varepsilon(x', \xi')|.$$

Then there exists $C_\varepsilon > 0$ such that

$$\forall u \in \mathscr{S}(\mathbb{R}^{n-1}), \ |\mathrm{op}(a_\varepsilon) u| \leq (M_\varepsilon + C_\varepsilon h^{1/2}) |u| \tag{85}$$

provided h is sufficiently small.

Proof. The method of proof is classical. We give here a proof in our context.
By the symbolic calculus, we have

$$\mathrm{op}(a_\varepsilon)^* \mathrm{op}(a_\varepsilon) = \mathrm{op}(|a_\varepsilon|^2) + \mathrm{op}(c)$$

where $c \in hS(1, g)$ and

$$\forall u \in \mathscr{S}(\mathbb{R}^{n-1}), \ |(\mathrm{op}(c) u|u)_0| \leq C_\varepsilon h |u|^2. \tag{86}$$

By assumption $M_\varepsilon^2 - |a_\varepsilon|^2 \geq 0$ and $M_\varepsilon^2 - |a_\varepsilon|^2 \in S(1, g)$. Proposition A.2 consequently gives some $C_\varepsilon > 0$ such that

$$\forall u \in \mathscr{S}(\mathbb{R}^{n-1}), \ M_\varepsilon^2 |u|^2 - (\mathrm{op}(|a_\varepsilon|^2) u|u)_0 + C_\varepsilon h |u|^2 \geq 0$$

thus by (86) we have also

$$M_\varepsilon^2 |u|^2 - |\mathrm{op}(a_\varepsilon) u|^2 + C_\varepsilon h |u|^2 \geq 0$$

which gives (85).

□

Lemma A.4. *Let $(a_\varepsilon)_{\varepsilon \in (0,1)}$ a family of $S(1, g)$. We assume that there exists $C > 0$ such that*

$$\forall \varepsilon \in (0, 1), \ \forall (x', \xi'), a_\varepsilon(0, \xi') = 0 \ \text{and} \ |\partial_{x'} a_\varepsilon(x', \xi')| \leq \frac{C}{\varepsilon}.$$

Then there exists $C_1 > 0$ and $h_\varepsilon, K_\varepsilon > 0$ such that for all $v_0 \in H^{1/2}(\mathbb{R}^{n-1})$ supported in $B_\kappa := \{x' \in \mathbb{R}^{n-1}, \ |x'| \leq \kappa\}$,

$$\forall h \leq h_\varepsilon, \ |\mathrm{op}(a_\varepsilon) z_0| \leq C_1 \frac{\kappa}{\varepsilon} |z_0| + K_\varepsilon h^{1/2} |v_0|_{1/2} \tag{87}$$

with $z_0 = \mathrm{op}(\lambda_+^{1/2}) v_0$.

Proof. Let χ and $\tilde{\chi}$ in $\mathscr{C}^\infty(\mathbb{R}^{n-1})$ such that $\chi = \tilde{\chi} = 1$ on B_κ, supported in $B_{2\kappa}$ and $\operatorname{supp} \chi \subset \{\tilde{\chi} = 1\}$. We have $\lambda_+^{1/2} \in S(\langle \xi' \rangle_\varepsilon^{1/2}, g)$ hence, by symbolic calculus,

$$\begin{aligned} \mathrm{op}(a_\varepsilon) z_0 &= \mathrm{op}(a_\varepsilon) \tilde{\chi} z_0 + \mathrm{op}(a_\varepsilon)(1 - \tilde{\chi}) \mathrm{op}(\lambda_+^{1/2}) \chi v_0 \\ &= \mathrm{op}(a_\varepsilon \tilde{\chi}) z_0 + \mathrm{op}(a_1) z_0 + \mathrm{op}(a_2) v_0, \end{aligned} \tag{88}$$

where $a_1 \in hS(1,g)$ and $a_2 \in hS(\langle\xi'\rangle_\varepsilon^{1/2}, g)$ (since the asymptotic expansion of a_2 is null).
We have, since $a_1 \in hS(1,g)$,

$$|\operatorname{op}(a_1)z_0| \le C_\varepsilon h|z_0|. \tag{89}$$

One has the estimate

$$\sup_{(x',\xi')} |a_\varepsilon(x',\xi')\chi(x')| = \sup_{(x',\xi')} \left|\chi(x')x'\int_0^1 \partial_{x'}a_\varepsilon(tx',\xi')dt\right| \le 2C\frac{\kappa}{\varepsilon} \tag{90}$$

and $a_\varepsilon\chi \in S(1,g)$. We can apply Lemma A.3 and get, by (90),

$$|\operatorname{op}(a_\varepsilon\chi)z_0| \le (2C\kappa/\varepsilon + C_\varepsilon h^{1/2})|z_0|. \tag{91}$$

On the oher hand, by symbolic calculus, we have

$$\operatorname{op}(\langle\xi'\rangle^{-1/2})\operatorname{op}(a_2) = \operatorname{op}(a_3)$$

where $a_3 \in hS(\langle\xi'\rangle_\varepsilon^{1/2}\langle\xi'\rangle^{-1/2}, g) \subset hS(1,g)$. Consequently,

$$|\operatorname{op}(a_2)v_0| = |\operatorname{op}(a_3)v_0|_{1/2} \le C_\varepsilon h|v_0|_{1/2}. \tag{92}$$

Finally, we have

$$|z_0| = |\operatorname{op}(\lambda_+^{1/2})v_0| \le C_\varepsilon|v_0|_{1/2}. \tag{93}$$

Following (88), (89), (91), (92) and (93), we get the estimate (87). □

B Symbol reduction and roots properties

We recall that

$$p_\varphi(x,\xi) = \xi_n^2 + 2i\partial_{x_n}\varphi\xi_n + q_2(x,\xi') + 2iq_1(x,\xi')$$

where $q_2(x,\xi') = -(\partial_{x_n}\varphi(x))^2 + r(x,\xi') - r(x,\partial_{x'}\varphi(x))$, $q_1(x,\xi') = \tilde{r}(x,\xi',\partial_{x'}\varphi(x))$ and

$$\mu(x,\xi') = q_2(x,\xi') + \frac{q_1(x,\xi')^2}{(\partial_{x_n}\varphi(x))^2}.$$

Lemma B.1.

- – *If $\mu(x,\xi') < 0$, the roots of of $p_\varphi(x,\xi',\xi_n)$ with respect to ξ_n have negative imaginary parts.*
  – *On the other hand, if $\mu(x,\xi') > -(\partial_{x_n}\varphi(x))^2$ the two roots of $p(x,\xi',\xi_n)$ with respect to ξ_n have different imaginary parts.*
  If we denote by $\rho_1(x,\xi')$ and $\rho_2(x,\xi')$ the roots such that $\operatorname{Im}(\rho_1(x,\xi')) > \operatorname{Im}(\rho_2(x,\xi'))$ then
  $$\exists C>0,\ \delta>0 \text{ such that } |\xi'| \ge C \Rightarrow \operatorname{Im}(\rho_1(x,\xi')) \ge \delta|\xi'|,$$
  $$\operatorname{Im}(\rho_1(x,\xi')) > -\partial_{x_n}\varphi(x) > \operatorname{Im}(\rho_2(x,\xi')). \tag{94}$$
  Moreover, if $\alpha > 0$ there exists $\delta > 0$ such that for all $(x,\xi') \in \mathbb{R}^{n-1}\times\mathbb{R}^{n-1}$ satisfying $\mu(x,\xi') \ge -(1-\alpha)(\partial_{x_n}\varphi(x))^2$,
  $$|\rho_1(x,\xi') + i\partial_{x_n}\varphi(x)| \ge \delta\langle\xi'\rangle \text{ and } |\operatorname{Im}\rho_2(x,\xi')| \ge \delta\langle\xi'\rangle. \tag{95}$$

- *If $r(0,\xi') = |\xi'|^2$, $|\partial_{x'}\varphi(0)| \le \varepsilon\partial_{x_n}\varphi(0)$ and, for some $\alpha > 0$, $\mu(0,\xi') \ge -(1-\alpha)(\partial_{x_n}\varphi(0))^2$ then, there exists $\delta > 0$ such that*
  $$|\xi'| \ge \delta\partial_{x_n}\varphi(0) \tag{96}$$

and, for sufficiently small ε, the following formula is valid

$$\rho_1(0,\xi') + i\partial_{x_n}\varphi(0) = i\partial_{x_n}\varphi(0)\rho\left(\frac{\xi'}{\partial_{x_n}\varphi(0)}\right) \tag{97}$$

where, setting $V := \frac{\partial_{x'}\varphi(0)}{\partial_{x_n}\varphi(0)}$, we have noted

$$\rho(\eta') := \sqrt{|\eta'|^2 - |V|^2 + 2i\eta'.V}\ .$$

Moreover, one has the following estimates

$$\begin{aligned} &\forall|\eta'| \geq \varepsilon, && |\eta'| + \varepsilon \geq \mathrm{Re}(\rho(\eta')) \geq |\eta'| - \varepsilon, \\ &\forall|\eta'| \geq 2\varepsilon, && |\mathrm{Im}(\rho(\eta'))| \leq 2\varepsilon. \end{aligned} \tag{98}$$

Proof. The first point is proved in [23, Lemme 3] using some geometric transformation. For the reader's convenience, we however give another elementary proof.
For simplicity, we do not write the variables (x,ξ') and we define the two roots ρ_1 and ρ_2 such that $\mathrm{Im}\,\rho_1 \geq \mathrm{Im}\,\rho_2$. Using equation $p_\varphi = 0$, we have

$$\rho_1 + \rho_2 = -2i\partial_{x_n}\varphi$$

and consequently, there exists $a, b \in \mathbb{R}$ such that

$$\rho_1 = a + ib, \quad \rho_2 = -a - i(2\partial_{x_n}\varphi + b).$$

It is sufficient to show that $b < 0$. Moreover, since $\rho_1\rho_2 = q_2 + 2iq_1$ and q_1, q_2 are real-valued, one gets

$$q_2 = -a^2 + b(2\partial_{x_n}\varphi + b), \quad q_1 = -a(2\partial_{x_n}\varphi + b)$$

and consequently, if $b \geq 0$,

$$q_2 + \frac{q_1^2}{(\partial_{x_n}\varphi)^2} = a^2\left(\left(2 + \frac{b}{\partial_{x_n}\varphi}\right)^2 - 1\right) + b(2\partial_{x_n}\varphi + b) \geq 0;$$

a contradiction with our assumption.
We now prove that ρ_1 and ρ_2 have different imaginary parts if $\mu(x,\xi') > -(\partial_{x_n}\varphi(x))^2$. Assuming that $\mathrm{Im}\,\rho_1 = \mathrm{Im}\,\rho_2$, by equation $p_\varphi = 0$ there also exists $a \in \mathbb{R}$ such that

$$\rho_1 = a - i\partial_{x_n}\varphi \text{ and } \rho_2 = -a - i\partial_{x_n}\varphi.$$

Thus $q_2 + 2iq_1 = \rho_1\rho_2 = -(\partial_{x_n}\varphi)^2 - a^2$ hence $q_1 = 0$ and $\mu = -(\partial_{x_n}\varphi)^2 - a^2 \leq -(\partial_{x_n}\varphi)^2$; a contradiction with the assumption.
Consequently, since $\rho_1 + \rho_2 = -2i\partial_{x_n}\varphi$ and $\mathrm{Im}\,\rho_1 > \mathrm{Im}\,\rho_2$, we obtain (94).

Moreover, we have

$$q_2(x,\xi') = r(x,\xi') + O(1) \text{ and } q_1(x,\xi') = O(\langle\xi'\rangle).$$

It is easy to deduce that

$$\rho_1(x,\xi') = i\sqrt{r(x,\xi')} + O(1) \text{ and } \rho_2(x,\xi') = -i\sqrt{r(x,\xi')} + O(1).$$

Hence for $|\xi'| > C$ where C is large enough, we have

$$\mathrm{Im}(\rho_1) \geq \delta|\xi'|,\ |\rho_1(x,\xi') + i\partial_{x_n}\varphi(x)| \geq \delta|\xi'| \text{ and } |\mathrm{Im}\,\rho_2(x,\xi')| \geq \delta|\xi'|$$

where $\delta > 0$ is sufficiently small. But we have already proved that $\rho_1(x,\xi') + i\partial_{x_n}\varphi(x) \neq 0$ and $\mathrm{Im}\,\rho_2(x,\xi') \neq 0$ thus, by compactness argument on $|\xi'| \leq C$, we get (95).

If $r(0,\xi') = |\xi'|^2$ then

$$\begin{aligned} p_\varphi(0,\xi',\xi_n) &= (\xi_n + i\partial_{x_n}\varphi(0))^2 + (\xi' + i\partial_{x'}\varphi(0)).(\xi' + i\partial_{x'}\varphi(0)) \\ &= (\xi_n + i\partial_{x_n}\varphi(0))^2 + |\xi'|^2 - |\partial_{x'}\varphi(0)|^2 + 2i\xi'.\partial_{x'}\varphi(0). \end{aligned}$$

On the other hand, if $\mu(0,\xi') \geq -(1-\alpha)(\partial_{x_n}\varphi(0))^2$, one has, writing $\eta' = \frac{\xi'}{\partial_{x_n}\varphi(0)}$,

$$|\eta'|^2 - |V|^2 \geq |\eta'.V|^2 + \alpha \quad \text{so that} \quad |\eta'| \geq \sqrt{\alpha}.$$

This proves (96).
Hence, for sufficiently small ε and since $|\partial_{x'}\varphi(0)| \leq \varepsilon\partial_{x_n}\varphi(0)$, we get

$$|\xi'|^2 - |\partial_{x'}\varphi(0)|^2 + 2i\xi'.\partial_{x'}\varphi(0) \notin \mathbb{R}_-$$

and we consequently deduce

$$\rho_1(0,\xi') + i\partial_{x_n}\varphi(0) = i\sqrt{|\xi'|^2 - |\partial_{x'}\varphi(0)|^2 + 2i\xi'.\partial_{x'}\varphi(0)},$$

which immediately give (97).
Using that, for any $z \in \mathbb{C}$ with $\operatorname{Re}(z) \geq 0$,

$$\operatorname{Re}(\sqrt{z}) \geq \sqrt{\operatorname{Re}(z)} \text{ and } 2\operatorname{Re}(\sqrt{z})\operatorname{Im}(\sqrt{z}) = \operatorname{Im}(z),$$

we finally have, for $|\eta'| \geq \varepsilon$ and since $|V| \leq \varepsilon$,

$$\operatorname{Re}(\rho(\eta')) \geq \sqrt{|\eta'|^2 - |V|^2} \geq |\eta'| - |V| \geq |\eta'| - \varepsilon$$

and, for $|\eta'| \geq 2\varepsilon$,

$$|\operatorname{Im}(\rho(\eta'))| \leq \frac{|V||\eta'|}{|\eta'| - \varepsilon} \leq \varepsilon\frac{|\eta'|}{|\eta'| - 1/2|\eta'|} \leq 2\varepsilon.$$

On the other hand, one has

$$|\operatorname{Re}(\rho(\eta'))| \leq \left((|\eta'|^2 - |V|^2)^2 + 4(\eta'.V)^2\right)^{1/4} \leq (|\eta'|^2 + |V|^2)^{1/2} \leq |\eta'| + |V|$$

which concludes the proof since $|V| \leq \varepsilon$. □

Lemma B.2. *Let $p(x,\xi)$ a $\mathscr{C}^\infty$ positive definite quadratic form in $T^*(U)$ where U is a neighborhood of $0 \in \mathbb{R}^n$.*
Let f a smooth function defined in U satisfying $f(0) = 0$ and $df \neq 0$.
Then there exists a change of variables such that, in the new variables (y,η), we have locally near 0

$f(x) > 0 \Leftrightarrow y_n > 0$ *and* $f(x) = 0 \Leftrightarrow y_n = 0$,

$q(y,\eta) = \eta_n^2 + r(y,\eta')$ *if* $\eta = (\eta', \eta_n)$,

with q the symbol p written in the new variables.
Moreover, if k a smooth function defined on a neighborhood of 0 in the submanifold $S := f^{-1}(\{0\})$ such that $k(0) = 0$ and $dk(0) \in T_0^*S \setminus \{0\}$ we can choose the new variables (y',η') such that

($k(x) > 0$ *on* $f(x) = 0 \Leftrightarrow y_1 > 0$ *on* $y_n = 0$) *and* ($k(x) = 0$ *on* $f(x) = 0 \Leftrightarrow y_1 = 0$ *on* $y_n = 0$),

$r(0,\eta') = |\eta'|^2$.

Proof. It is classical (see [14, Corollary C.5.3] for instance) that we can find change of variables such that

$$(f(x) > 0 \Leftrightarrow y_n > 0), \quad (f(x) = 0 \Leftrightarrow y_n = 0) \text{ and } q(y,\eta) = \eta_n^2 + r(y,\eta')$$

with r a homogeneous polynomial of order 2 in η'. Moreover, using Taylor formula, one may write

$$r(y,\eta') = r(y',0,\eta') + y_n r_1(y,\eta').$$

By the same method, we can choose variables $z' = (z_1, z'')$ on S such that, denoting s the function r written in the variables (z', ζ'),

$$(k(y') > 0 \Leftrightarrow z_1 > 0), \quad (k(y') = 0 \Leftrightarrow z_1 = 0) \text{ and } s(z', 0, \zeta') = \zeta_1^2 + s_1(z', \zeta''),$$

where we have set $\zeta' = (\zeta_1, \zeta'')$.
Finally, by a linear change of variables in z'' (which does not pertub the other term), we can write $s_1(0, \zeta'') = |\zeta''|^2$ and consequently get

$$s(0, \zeta') = |\zeta'|^2.$$

□

Remark 7.

- A $\mathscr{C}^\infty$ positive definite quadratic form in $T^*(U)$ is a $\mathscr{C}^\infty$ map such that for all $x \in U$, $p(x, .)$ is a positive definite quadratic form.
- We note that this Lemma takes a more standard form if k can be defined in a neighborhood of $0 \in \mathbb{R}^n$. Indeed, one has the following result.

  If f and k are smooth function defined in a neighborhood of 0, satisfying $f(0) = k(0) = 0$ and such that $df(0)$ and $dk(0)$ are independent, then there exists a change of variables such that, in the new variable (y, η) and locally near 0,

  $$\begin{aligned} &f(x) > 0 \Leftrightarrow y_n > 0, \\ &k(x) > 0 \text{ on } f(x) = 0 \Leftrightarrow y_1 > 0 \text{ on } y_n = 0, \\ &q(y, \eta) = \eta_n^2 + r(y, \eta') \text{ where } \eta = (\eta', \eta_n), \\ &r(0, \eta') = |\eta'|^2. \end{aligned}$$

C A norm estimate

Lemma C.1. *Let $\rho \in (0,1)$, $a \in \mathscr{C}^\rho(\mathbb{R}^{n-1})$ such that $a_{|x_1=0} = 0$ and χ a smooth function supported in the unit ball B_1. Then there exists $C > 0$ such that, for any $\rho' \leq \rho$ and any $\lambda \in (0,1)$,*

$$\|a\chi(./\lambda)\|_{\mathscr{C}^{\rho'}} \leq C\lambda^{\rho-\rho'}\|a\|_{\mathscr{C}^\rho}$$

where, for any $x \in \mathbb{R}^{n-1}$, $\chi(./\lambda)(x) = \chi(x/\lambda)$.

Proof. We first estimate $|(a\chi(./\lambda))(x) - (a\chi(./\lambda))(y)|$ depending on the positions of x and y.

- If $x, y \notin B_\lambda$, one clearly has

  $$|(a\chi(./\lambda))(x) - (a\chi(./\lambda))(y)| \leq C\lambda^{\rho-\rho'}\|a\|_{\mathscr{C}^\rho}\ |x-y|^{\rho'}.$$

- If $x, y \in B_\lambda$, one has

  $$(a\chi(./\lambda))(x) - (a\chi(./\lambda))(y) = a(x)\left(\chi\left(\frac{x}{\lambda}\right) - \chi\left(\frac{y}{\lambda}\right)\right) + (a(x) - a(y))\chi\left(\frac{y}{\lambda}\right)$$

  which gives, since $|x-y| \leq 2\lambda$ and χ is smooth,

  $$\begin{aligned} |(a\chi(./\lambda))(x) - (a\chi(./\lambda))(y)| &\leq C\left(|a(x)|\frac{|x-y|}{\lambda} + \|a\|_{\mathscr{C}^\rho}|x-y|^\rho\right) \\ &\leq C\left(|a(x)|\frac{|x-y|^{\rho'}}{\lambda^{\rho'}} + \lambda^{\rho-\rho'}\|a\|_{\mathscr{C}^\rho}|x-y|^{\rho'}\right). \end{aligned}$$

Using now that $a_{|x_1=0} = 0$, one has moreover

$$|a(x)| \leq |x_1|^\rho \|a\|_{\mathscr{C}\rho} \ \leq C\lambda^\rho \ \|a\|_{\mathscr{C}^\rho}$$

and finally gets

$$\forall x, y \in B_\lambda,\ |(a\chi(./\lambda))(x) - (a\chi(./\lambda))(y)| \leq C\lambda^{\rho-\rho'}\|a\|_{\mathscr{C}^\rho}\ |x-y|^{\rho'}.$$

- If now $x \in B_\lambda$, $y \notin B_\lambda$, one has

$$|(a\chi(./\lambda))(x) - (a\chi(./\lambda))(y)| = |(a\chi(./\lambda))(x)| = |(a\chi(./\lambda))(x) - (a\chi(./\lambda))(t)|$$

  where $t \in [x, y]$ and $|t| = \lambda$. Using the second case, we also get

$$|(a\chi(./\lambda))(x) - (a\chi(./\lambda))(y)| \leq C\lambda^{\rho-\rho'}\|a\|_{\mathscr{C}^\rho}\ |x-y|^{\rho'}$$

  since $|x - t| \leq |x - y|$.

Finally, it is obvious that

$$\|a\chi(./\lambda)\|_{L^\infty} \leq \|\chi\|_{L^\infty}\|a\|_{L^\infty(B_\lambda)} \leq C\|a\|_{L^\infty(B_\lambda)}$$

and since, for any $x \in B_\lambda$, $|a(x)| \leq C\lambda^\rho\|a\|_{\mathscr{C}^\rho}$,

$$\|a\chi(./\lambda)\|_{L^\infty} \leq C\lambda^{\rho-\rho'}\|a\|_{\mathscr{C}^\rho}.$$

The proof is complete. □

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